

Analysis of Stochastic Switched Systems with Application to Networked Control Under Jamming Attacks

Ahmet Cetinkaya, Hideaki Ishii, and Tomohisa Hayakawa

Abstract—We investigate the stability problem for discrete-time stochastic switched linear systems. We consider scenarios where information about the switching patterns and the probability of switches are not available for analysis. Our setup allows us to utilize switched systems to study the networked control problem under packet losses due to malicious attacks where the attacker's strategy is not known a priori. We obtain almost sure asymptotic stability conditions and show that the stability of a switched system can be assessed by examining the solution to a linear programming problem. The objective function in this linear programming problem depends on the dynamics of a new system that describes the evolution of the switched system's state at every few steps. Furthermore, the coefficients of the constraints in the problem depend on lower- and upper-bounds on the average number of times each mode becomes active in the long run. We discuss the computational complexity of solving this linear programming problem. We also show that the optimal solution value can be obtained by solving an alternative linear programming problem that has fewer variables but requires additional computation for generating the coefficients for the objective function. We demonstrate the efficacy of our results in the analysis of two networked control problems where communication channels face packet losses due to jamming attacks.

Index Terms—Stochastic switched systems, stability analysis, linear programming, networked control systems, jamming, packet losses.

I. INTRODUCTION

A broad range of complex dynamics from finance, robotics, automotive systems, power systems, and networked control fields are accurately characterized through switched systems. The stability problem for switched systems has been extensively studied and various stability conditions have been derived based on the properties of subsystem dynamics and the mode signal of a switched system [1], [2].

Stability of switched systems with arbitrary switching was explored in a number of studies including [3], [4], where researchers provided conditions under which a switched system remains stable for all possible mode switching scenarios. Several other works considered restrictions on the mode signal. In particular, [5] and [6] utilized dwell-time and average-dwell-time notions to characterize conditions on the frequency of mode switching between stable subsystems to guarantee the overall stability of a switched system. The works [7]

and [8] extended the notion of average-dwell-time to deal with switched systems that are composed of both stable and unstable subsystems. In addition, [9] investigated the problem of designing state-dependent switching rules to guarantee stability. The stability of switched systems with stochastic mode signals have also been studied in several works (see [10], [11] and the references therein). In particular, Markov jump systems, for which the mode signal is modeled using Markov chains, attracted considerable attention [12], [13]. Moreover, [14], [15] recently introduced and utilized “dual switching” notion to characterize both stochastic and deterministic effects in the switching of the mode signal.

In the literature of stochastic switched systems, stability results often rely on statistical information on the mode signal. For instance, the probability of mode switches and the stationary distributions associated with the modes are commonly used in the analysis. However, in certain problem settings, such information is not available and new analysis approaches are needed.

In this paper, we investigate a stability problem for discrete-time stochastic switched linear systems and consider the case where information about the mode switching probabilities or the stationary distributions for the modes are not available for analysis. Our motivation comes from the networked control problem under malicious attacks. In networked control, transmissions over communication channels between a plant and a controller may fail due to malicious attacks [16]–[18]. For instance, there may be denial-of-service (DoS) and jamming attacks [19]–[21] or packet drops by compromised routers [22], [23].

The formulation of the stochastic switched system problem studied in this paper is based on the approach proposed in our recent work [18]; see also [24]–[27] for more related results. There, a networked control system under malicious jamming attacks is considered, where the jamming causes losses of data during packet transmissions due to strong interferences. Such a system can be modeled as a switched system where the dynamics of the normal operation and the dynamics under jamming attacks are represented with different modes. The main challenge in the analysis of such a switched system is that the mode switching is affected by the attacker's actions. The transmission failure instants and the probability of transmission failures are not available for analysis, because the attacker's actions cannot be known beforehand. In [18], we proposed a stochastic model regarding the timings of jamming using only the asymptotic tail probabilities of average transmission failures. The model is general enough to allow us to also describe nonmalicious issues in networked control such as packet losses due to network congestion and random communication errors. Typically, such losses are represented

A. Cetinkaya and H. Ishii are with the Department of Computer Science, Tokyo Institute of Technology, Yokohama, 226-8502, Japan. ahmet@sc.dis.titech.ac.jp, ishii@c.titech.ac.jp

T. Hayakawa is with the Department of Systems and Control Engineering, Tokyo Institute of Technology, Tokyo 152-8552, Japan. hayakawa@mei.titech.ac.jp

This work was supported in part by the JST CREST Grant No. JP-MJCR15K3 and by JSPS under Grant-in-Aid for Scientific Research Grant No. 15H04020.

using Bernoulli processes [28], [29] and Markov processes [30]–[32] with known statistical properties. Under this loss model, in [18], we developed results for analyzing almost sure asymptotic stability of the overall networked system based on solving Lyapunov-type linear matrix inequalities.

In this paper, we generalize the class of systems in the networked control problem under malicious jamming attacks mentioned above, and develop an approach for stability analysis of stochastic switched systems. Furthermore, we focus on improving the efficiency of the stability analysis. Specifically, in our approach, stability is assessed through a condition on the optimal value of the objective function of a linear programming problem. In this problem, the coefficients in the objective function are obtained based on the subsystems' dynamics. Moreover, the coefficients of the constraints in the problem are obtained by utilizing available lower- and upper-bounds on the percentage of the time each mode is active in the long run. Our stability analysis approach differs from those in [33]–[36], where linear programming is used for constructing Lyapunov functions. In this paper, we utilize linear programming to identify the worst mode switching scenario in terms of stability, and assess the stability of the system under that scenario. A key property of our approach is that the probability of mode switches or the possible switching patterns need not be known in order to check stability.

To formulate our linear programming problem for stability analysis, we first construct a new system that describes the evolution of the switched system's state at every $h \in \mathbb{N}$ steps. For a given switched system with M modes, this new system is composed of M^h modes, each of which is identified by a sequence of h numbers indicating a progression of modes in the original switched system. We observe that the optimal objective function value for our linear programming problem may be positive when h is small and be negative for sufficiently large h . This is because, with large h , stability/instability properties of more switching patterns are taken into account. This indicates the advantage of solving the linear programming problem for large values of h . To deal with the computational difficulties when h is large, we utilize an alternative linear programming problem that has fewer variables than the original problem but shares the same optimal objective function value. Even though the calculation of the coefficients in the alternative problem takes additional time, the solution is obtained faster compared to the original problem.

The idea of investigating stability by means of studying a new system that describes the state's evolution at every few steps has previously been employed in [37]–[40]. In [37] and [38], Markov jump systems are explored. The results there utilize the transition probabilities between the sequences composed of the possible values of a Markov chain as well as the stationary distributions of the sequences. Those statistical information are not available in our problem setting. On the other hand, in [39] and [40], researchers explored the stability problem under constrained switching rules. In those studies all possible switching patterns are identified through graphs that indicate the constraints in the switching. In our problem setting, information about the possible switching patterns is

not available, and moreover, we consider scenarios where switching is allowed between all pairs of modes. For this particular setting, the stability results in [39], [40] require all subsystems to be stable. Therefore, they are not applicable for our problem since we allow both stable and unstable subsystem dynamics. In this paper, by using the lower- and upper-bounds on the percentage of the time each mode is active in the long run, we show that the asymptotic ratio of the occurrences of each mode sequence satisfies certain inequalities. These inequalities form a basis for the linear programming problems that we use for stability analysis.

We show in the paper that our linear programming problems always possess solutions. Furthermore, the negativity of the optimal objective function values implies stability of the switched system, whenever the limits corresponding to the the asymptotic ratios of the occurrences of sequences with length h exist. We show that this limit requirement is satisfied for a class of mode signals that allow us to capture several scenarios in the networked control problem under random and malicious packet transmission failures. These scenarios include the periodic attacks discussed in [16] as well as random packet losses on channels described with Markov models explored in [41], [42].

We apply our stability results in two networked control problems. In the first problem, the plant and the remotely located controller are assumed to exchange information packets over delay-free communication channels. These channels are subject to packet losses due to random communication errors or malicious attacks. We represent the dynamics associated with the networked control system as a switched system with two modes associated with the success and failure states of the packet exchange attempts. We then investigate the stability by utilizing an upper-bound on the long-run average number of packet exchange failures. We observe that in certain cases our linear programming-based stability condition obtained in this paper is less conservative than the stability condition from our earlier work [18], where we used a Lyapunov-like function approach. In our second problem, we consider the situation where two separate communication channels are used for transmission of the control input packets from the controller to the plant. We assume that one of the channels faces no delay, but the other one faces one-step delay in transmissions. Both channels are assumed to be subject to attacks. We show that the networked control system in this setup can be described as a switched system with three modes. We explore the stability by utilizing the available bounds for packet transmission failure ratios associated with the communication channels.

The rest of the paper is organized as follows. In Section II, we explain the switched system dynamics and obtain stability conditions by utilizing bounds on the long run average number of times each mode becomes active. In Section III, we present our approach for checking stability conditions by means of solving linear programming problems. Moreover, in Section IV, we discuss the application of our results in the problem of networked control under malicious attacks. We present numerical examples in Section V. Finally, in Section VI, we conclude the paper.

We note that part of the results in Sections II and III

appeared in our preliminary report [43] for the special case of two modes in the context of a networked control problem. Here, we provide complete proofs and more detailed discussions for the general modes case.

We use a fairly standard notation in the paper. Specifically, \mathbb{N} and \mathbb{N}_0 respectively denote positive and nonnegative integers. A finite-length sequence of ordered elements q_1, q_2, \dots, q_h is represented by $q = (q_1, q_2, \dots, q_h)$. We use $\lfloor \cdot \rfloor$ to denote the largest integer that is smaller than or equal to its real argument. The notation $\mathbb{P}[\cdot]$ denotes the probability on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Furthermore, we utilize $\mathbb{1}[E] : \Omega \rightarrow \{0, 1\}$ for the indicator of the event $E \in \mathcal{F}$, that is, $\mathbb{1}[E](\omega) = 1$ for $\omega \in E$, and $\mathbb{1}[E](\omega) = 0$ for $\omega \notin E$.

II. STOCHASTIC SWITCHED SYSTEM STABILITY ANALYSIS

In this section we first describe the dynamics of a stochastic switched system. We then discuss the stability problem and provide sufficient almost sure asymptotic stability conditions for stochastic switched systems.

A. Switched System Dynamics

Consider the discrete-time switched linear system with $M \in \mathbb{N}$ modes described by

$$x(t+1) = A_{r(t)}x(t), \quad x(0) = x_0, r(0) = r_0, t \in \mathbb{N}_0, \quad (1)$$

where $x(t) \in \mathbb{R}^n$ denotes the state vector, $\{r(t) \in \{1, \dots, M\}\}_{t \in \mathbb{N}_0}$ is the mode signal, and $A_s \in \mathbb{R}^{n \times n}$, $s \in \{1, \dots, M\}$, represent the system matrices for each mode. We use $\mathcal{M} \triangleq \{1, \dots, M\}$ to denote the set of modes.

The mode signal $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ is assumed to be a stochastic process that satisfies the following assumption.

Assumption 2.1: There exist scalars $\underline{\rho}_s, \bar{\rho}_s \in [0, 1]$, $s \in \mathcal{M}$, such that

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \geq \underline{\rho}_s, \quad (2)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \leq \bar{\rho}_s, \quad s \in \mathcal{M}, \quad (3)$$

almost surely.

In Assumption 2.1, the scalars $\underline{\rho}_s$ and $\bar{\rho}_s$ respectively represent lower- and upper-bounds on the long-run average number of times mode s is active. If no information is available on the long-run average for mode s , the scalars $\underline{\rho}_s$ and $\bar{\rho}_s$ can be selected as $\underline{\rho}_s = 0$ and $\bar{\rho}_s = 1$, since (2) and (3) are trivially satisfied with those values.

Note that Assumption 2.1 allows the mode signal $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ to be generated randomly according to a probability distribution or in a deterministic fashion. For instance, $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ can be an irreducible Markov chain with transition probability matrix $P \in \mathbb{R}^{M \times M}$ and invariant distribution $\pi \in [0, 1]^M$. In that case $\underline{\rho}_s$ and $\bar{\rho}_s$ would be scalars that satisfy $\underline{\rho}_s \leq \pi_s \leq \bar{\rho}_s$, $s \in \mathcal{M}$. On the other hand, $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ can also represent a deterministically generated switching sequence. For example, a particular mode switching pattern of length $T \in \mathbb{N}$ can be repeated to create

a periodic switching sequence. In that case, $\underline{\rho}_s$ and $\bar{\rho}_s$ would correspond to lower- and upper-bounds on the ratio of the number of times mode s is active in the T -length switching pattern. In the case where $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ is deterministic, the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = q]$, $q \in \mathcal{M}$, when they exist, correspond to the “discrete event rate” utilized by [44], [45] for deterministic systems.

In our stability analysis in the following section, we utilize the scalars $\underline{\rho}_s$ and $\bar{\rho}_s$ satisfying (2) and (3) instead of transition probabilities or switching patterns. This is because our goal is to consider scenarios where precise information about how mode switches occur is not available for analysis. In Section IV, we model networked control systems as switched systems, where the mode signals represent the state of certain communication channels that face malicious attacks. There, $\underline{\rho}_s$ and $\bar{\rho}_s$ are characterized based on the information about the average number of times transmissions fail due to attacks in the long run.

B. Stability Analysis

In this section, we explore the stability of the switched system (1), where the mode signal satisfies Assumption 2.1. We use the stochastic stability notion of *almost sure asymptotic stability* in our analysis.

Definition 2.1: The zero solution $x(t) \equiv 0$ of the stochastic system (1) is *almost surely stable* if for each $\epsilon > 0$ and $\bar{p} > 0$, there exists $\delta = \delta(\epsilon, \bar{p}) > 0$ such that if $\|x_0\|_2 < \delta$, then

$$\mathbb{P}[\sup_{t \in \mathbb{N}_0} \|x(t)\|_2 > \epsilon] < \bar{p}, \quad (4)$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Moreover, the zero solution $x(t) \equiv 0$ is *asymptotically stable almost surely* if it is almost surely stable and

$$\mathbb{P}[\lim_{t \rightarrow \infty} \|x(t)\|_2 = 0] = 1. \quad (5)$$

Stability of discrete-time switched systems have been explored in many studies under different assumptions on the mode signal. For instance, in several works (see [2] and the references therein) researchers explore stability of systems with a form similar to (1) under arbitrary switching. A necessary condition for stability under arbitrary switching is individual stability of each mode (A_1, A_2, \dots, A_M need to be Schur matrices). In our problem setting we allow some of the modes to be unstable.

In addition to stability analysis under arbitrary switching, researchers also explore stability of systems similar to (1) for the case where $r(\cdot)$ is a Markov process (see, e.g., [12], [13], [37]). The stability analysis in those studies rely on transition probabilities and stationary distributions associated with the Markov process that characterizes the switching sequence. Note that in our case, $r(\cdot)$ need not be a Markov process. Furthermore, to account for the uncertainty in generation of the mode signal, we assume that statistical information concerning transition probabilities and stationary distributions is not available in our problem setting. Hence, the stability results reported in the above-mentioned literature are not applicable to the present problem.

In our stability analysis of the switched system, we follow the approach in [37], [39] and investigate the evolution of the system's state at every $h \in \mathbb{N}$ steps. First, let \mathcal{M}^h denote the set of sequences of length h with entries in \mathcal{M} , that is,

$$\mathcal{M}^h \triangleq \{(q_1, q_2, \dots, q_h) : q_j \in \mathcal{M}, j \in \{1, \dots, h\}\}.$$

With this definition, q_i (i th entry of a sequence q) represents a mode in the set of modes \mathcal{M} . Now, let $\{\bar{r}(i) \in \mathcal{M}^h\}_{i \in \mathbb{N}_0}$ be a sequence-valued process defined by

$$\bar{r}(i) \triangleq (r(ih), r(ih+1), \dots, r((i+1)h-1)), \quad i \in \mathbb{N}_0. \quad (6)$$

It then follows that the state evaluated at every h steps is described by

$$x((i+1)h) = \Gamma_{\bar{r}(i)} x(ih), \quad i \in \mathbb{N}_0, \quad (7)$$

where

$$\Gamma_q \triangleq A_{q_h} A_{q_{h-1}} \cdots A_{q_1}, \quad q \in \mathcal{M}^h. \quad (8)$$

The dynamical system (7) is a switched system with M^h modes. Each mode of this system is identified by a sequence of h numbers from \mathcal{M} representing the modes of the original switched system (1).

Now, let $c_s : \mathcal{M}^h \rightarrow \{0, \dots, h\}$ be defined by $c_s(q) \triangleq \sum_{j=1}^h \mathbb{1}[q_j = s], q \in \mathcal{M}^h, s \in \mathcal{M}$. With this definition, the number of entries with value s in the sequence $q \in \mathcal{M}^h$ is represented with $c_s(q)$. Note that c_s satisfies

$$\sum_{i=0}^{k-1} \sum_{q \in \mathcal{M}^h} c_s(q) \mathbb{1}[\bar{r}(i) = q] = \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}, \quad (9)$$

which establishes a key relation between the mode signal $r(\cdot)$ and the sequence-valued process $\bar{r}(\cdot)$.

In Lemma 2.2 below, we use (9) to obtain a relation between $\underline{\rho}_s, \bar{\rho}_s$ in Assumption 2.1 and the long-run average numbers of the occurrences of all sequences in \mathcal{M}^h . The long run average for a sequence $q \in \mathcal{M}^h$ is given by

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q],$$

whenever this limit exists, that is, $\frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ converges almost surely to a random variable as $k \rightarrow \infty$.

Lemma 2.2: Suppose $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ satisfies Assumption 2.1 with $\underline{\rho}_s, \bar{\rho}_s \in [0, 1], s \in \mathcal{M}$. If $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$, then we have

$$\sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \bar{\rho}_s, \quad (10)$$

$$\sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \geq \underline{\rho}_s, \quad (11)$$

for $s \in \mathcal{M}$, almost surely.

Proof: First, by (9),

$$\begin{aligned} & \sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \\ &= \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{k-1} \sum_{q \in \mathcal{M}^h} c_s(q) \mathbb{1}[\bar{r}(i) = q] \\ &= \lim_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s]. \end{aligned} \quad (12)$$

Here, we have

$$\begin{aligned} & \left\{ \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] : \bar{k} \geq k \right\} \\ & \subset \left\{ \frac{1}{k} \sum_{i=0}^{\bar{k}-1} \mathbb{1}[r(i) = s] : \bar{k} \geq k \right\}, \quad k \in \mathbb{N}, \end{aligned} \quad (13)$$

and hence

$$\sup_{\bar{k} \geq k} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] \leq \sup_{\bar{k} \geq k} \frac{1}{k} \sum_{i=0}^{\bar{k}-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}.$$

Therefore,

$$\limsup_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] \leq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s]. \quad (14)$$

As a result, (10) follows from (12), (3) and (14).

Similarly, it follows from (13) that

$$\inf_{\bar{k} \geq k} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] \geq \inf_{\bar{k} \geq k} \frac{1}{k} \sum_{i=0}^{\bar{k}-1} \mathbb{1}[r(i) = s], \quad k \in \mathbb{N}.$$

Therefore,

$$\liminf_{k \rightarrow \infty} \frac{1}{kh} \sum_{i=0}^{kh-1} \mathbb{1}[r(i) = s] \geq \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[r(i) = s]. \quad (15)$$

Finally, (11) follows from (12), (2) and (15). \square

Next, we employ Lemma 2.2 to establish conditions for almost sure asymptotic stability. To this end, first, for a given matrix $N \in \mathbb{R}^{n \times n}$, let $\|N\|$ denote the induced matrix norm defined by

$$\|N\| \triangleq \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Nx\|}{\|x\|}, \quad (16)$$

where $\|\cdot\|$ on the right-hand side denotes a vector norm on \mathbb{R}^n . In the proof of the next result, we use the submultiplicativity property of induced matrix norms, i.e., $\|N_1 N_2\| \leq \|N_1\| \|N_2\|$ for $N_1, N_2 \in \mathbb{R}^{n \times n}$ (see Section 5.6 in [46]).

Theorem 2.3: Consider the switched system (1). Suppose that the mode signal $\{r(t) \in \{1, \dots, M\}\}_{t \in \mathbb{N}_0}$ satisfies Assumption 2.1 with $\bar{\rho}_s, \underline{\rho}_s \in [0, 1], s \in \mathcal{M}$, and $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$ for a given $h \in \mathbb{N}$. If there exist an induced matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0, 1)$ such that the inequality

$$\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q < 0, \quad (17)$$

holds with

$$\gamma_q \triangleq \begin{cases} \ln \|\Gamma_q\|, & \Gamma_q \neq 0, \\ \ln \varepsilon, & \Gamma_q = 0, \end{cases} \quad q \in \mathcal{M}^h, \quad (18)$$

for all $\rho_q \in [0, 1], q \in \mathcal{M}^h$, that satisfy

$$\sum_{q \in \mathcal{M}^h} \rho_q = 1, \quad (19)$$

$$\rho_s \leq \sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \rho_q \leq \bar{\rho}_s, \quad s \in \mathcal{M}, \quad (20)$$

then the zero solution $x(t) \equiv 0$ of the dynamical system (1) is asymptotically stable almost surely.

Proof: First, it follows from (7) that

$$\|x((k+1)h)\| = \|\Gamma_{\bar{r}(k)} x(kh)\| \leq \|\Gamma_{\bar{r}(k)}\| \|x(kh)\|,$$

and hence by the submultiplicativity property of the induced matrix norm $\|\cdot\|$, we have

$$\|x(kh)\| \leq \eta(k) \|x_0\|, \quad k \in \mathbb{N}_0, \quad (21)$$

where $\eta(k) \triangleq \prod_{i=0}^{k-1} \|\Gamma_{\bar{r}(i)}\|$. Now we define $\mu(k) \triangleq \sum_{i=0}^{k-1} \gamma_{\bar{r}(i)}$, $k \in \mathbb{N}_0$, where $\gamma_q, q \in \mathcal{M}^h$, are given by (18). It follows from (18) together with the definitions of $\eta(k)$ and $\mu(k)$ that $\eta(k) \leq e^{\mu(k)}$, $k \in \mathbb{N}_0$. Furthermore, we have

$$\mu(k) = \sum_{q \in \mathcal{M}^h} \gamma_q \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q], \quad k \in \mathbb{N}_0,$$

and as a result,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \mu(k) = \sum_{q \in \mathcal{M}^h} \gamma_q \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q], \quad (22)$$

almost surely. Here, note that $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \in [0, 1]$. Furthermore, since

$$\begin{aligned} & \sum_{q \in \mathcal{M}^h} \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{q \in \mathcal{M}^h} \mathbb{1}[\bar{r}(i) = q] = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} 1 = 1, \end{aligned}$$

it follows that $\rho_q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, satisfy (19). On the other hand, Lemma 2.2 implies (20) with $\rho_q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$. Now, since (17) is satisfied for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that (19), (20) hold, we have $\sum_{q \in \mathcal{M}^h} \gamma_q \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] < 0$. Hence, as a consequence of (22),

$$\lim_{k \rightarrow \infty} \frac{1}{k} \mu(k) < 0,$$

and $\lim_{k \rightarrow \infty} \mu(k) = -\infty$, almost surely. Since $\eta(k) \leq e^{\mu(k)}$, we obtain $\mathbb{P}[\lim_{k \rightarrow \infty} \eta(k) = 0] = 1$. Thus, for any $\epsilon > 0$, $\lim_{j \rightarrow \infty} \mathbb{P}[\sup_{k \geq j} \eta(k) > \epsilon] = 0$. Therefore, for any $\epsilon > 0$ and $\bar{p} > 0$, there exists a positive integer $N(\epsilon, \bar{p})$ such that

$$\mathbb{P}[\sup_{k \geq j} \eta(k) > \epsilon] < \bar{p}, \quad j \geq N(\epsilon, \bar{p}). \quad (23)$$

In what follows, we show almost sure stability of the switched system by using (21) and (23). First, we define $\phi \triangleq \max\{\|A\|, \|A+BK\|\}$ and $\mathcal{T}_k \triangleq \{kh, \dots, (k+1)h-1\}$, $k \in \mathbb{N}_0$. Using these definitions, we obtain $\|x(t+1)\| \leq \phi \|x(t)\|$, $t \in \mathcal{T}_k$, and as a result,

$$\max_{t \in \mathcal{T}_k} \|x(t)\| \leq \phi^h \|x(kh)\|, \quad k \in \mathbb{N}_0. \quad (24)$$

Now by (21) and (24),

$$\eta(k) \geq \|x(kh)\| \|x_0\|^{-1} \geq \max_{t \in \mathcal{T}_k} \|x(t)\| \phi^{-h} \|x_0\|^{-1}, \quad k \in \mathbb{N}_0.$$

Then it follows from (23) that for all $\epsilon > 0$ and $\bar{p} > 0$,

$$\begin{aligned} & \mathbb{P}[\sup_{k \geq j} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon \phi^h \|x_0\|] \\ &= \mathbb{P}[\sup_{k \geq j} \max_{t \in \mathcal{T}_k} \|x(t)\| \phi^{-h} \|x_0\|^{-1} > \epsilon] \\ &\leq \mathbb{P}[\sup_{k \geq j} \eta(k) > \epsilon] < \bar{p}, \quad j \geq N(\epsilon, \bar{p}). \end{aligned}$$

Now let $\delta_1 \triangleq \phi^{-h}$. Notice that if $\|x_0\| \leq \delta_1$, then $\phi^h \|x_0\| \leq 1$, and therefore, for all $j \geq N(\epsilon, \bar{p})$, we have

$$\begin{aligned} & \mathbb{P}[\sup_{k \geq j} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon] \\ &\leq \mathbb{P}[\sup_{k \geq j} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon \phi^h \|x_0\|] < \bar{p}. \end{aligned} \quad (25)$$

Furthermore, observe that for all $k \in \{0, 1, \dots, N(\epsilon, \bar{p}) - 1\}$, we have $\|x(kh)\| \leq \phi^k \|x_0\| \leq \phi^{N(\epsilon, \bar{p})-1} \|x_0\|$. Hence, as a result of (24),

$$\max_{t \in \mathcal{T}_k} \|x(t)\| \leq \phi^h \|x(kh)\| \leq \phi^{h+N(\epsilon, \bar{p})-1} \|x_0\|, \quad (26)$$

for all $k \in \{0, 1, \dots, N(\epsilon, \bar{p}) - 1\}$. Let $\delta_2(\epsilon, \bar{p}) \triangleq \epsilon \phi^{1-h-N(\epsilon, \bar{p})}$. Now, if $\|x_0\| \leq \delta_2(\epsilon, \bar{p})$, then by (26), $\max_{t \in \mathcal{T}_k} \|x(t)\| \leq \epsilon$, $k \in \{0, 1, \dots, N(\epsilon, \bar{p}) - 1\}$. Thus, if $\|x_0\| \leq \delta_2(\epsilon, \bar{p})$, then

$$\mathbb{P}[\max_{k \in \{0, 1, \dots, N(\epsilon, \bar{p})\}} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon] = 0. \quad (27)$$

Due to (25) and (27), for all $\epsilon > 0$, $\bar{p} > 0$, we have

$$\begin{aligned} & \mathbb{P}[\sup_{t \in \mathbb{N}_0} \|x(t)\| > \epsilon] = \mathbb{P}[\sup_{k \in \mathbb{N}_0} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon] \\ &= \mathbb{P}[\{\max_{k \in \{0, \dots, N(\epsilon, \bar{p})-1\}} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon\} \\ &\quad \cup \{\sup_{k \geq N(\epsilon, \bar{p})} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon\}] \\ &\leq \mathbb{P}[\max_{k \in \{0, \dots, N(\epsilon, \bar{p})-1\}} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon] \\ &\quad + \mathbb{P}[\sup_{k \geq N(\epsilon, \bar{p})} \max_{t \in \mathcal{T}_k} \|x(t)\| > \epsilon] < \bar{p}, \end{aligned} \quad (28)$$

whenever $\|x_0\| < \min(\delta_1, \delta_2(\epsilon, \bar{p}))$.

Now, by Corollary 5.4.5 of [46], there exist $c_1, c_2 > 0$ such that

$$c_1 \|x\| \leq \|x\|_2 \leq c_2 \|x\|, \quad x \in \mathbb{R}^n. \quad (29)$$

As a result, by using (28) and (29), we obtain that for all $\epsilon > 0$, $\bar{p} > 0$,

$$\mathbb{P}[\sup_{t \in \mathbb{N}_0} \|x(t)\|_2 > \epsilon] \leq \mathbb{P}[\sup_{t \in \mathbb{N}_0} \|x(t)\| > \frac{\epsilon}{c_2}] < \bar{p},$$

whenever $\|x_0\| < \min(\delta_1, \delta_2(\frac{\epsilon}{c_2}, \bar{p}))$. Now, since $\|x_0\| \leq \frac{\|x_0\|_2}{c_1}$, we have that for all $\epsilon > 0$, $\bar{p} > 0$, the inequality (4) holds whenever $\|x_0\|_2 < \delta(\epsilon, \bar{p}) \triangleq c_1 \min(\delta_1, \delta_2(\frac{\epsilon}{c_2}, \bar{p}))$, which implies almost sure stability.

Next, we show (5) to establish almost sure asymptotic stability of the zero solution. In this regard, first notice that $\mathbb{P}[\lim_{k \rightarrow \infty} \eta(k) = 0] = 1$. By using (21),

we get $\mathbb{P}[\lim_{k \rightarrow \infty} \|x(kh)\| = 0] = 1$, which implies $\mathbb{P}[\lim_{t \rightarrow \infty} \|x(t)\| = 0] = 1$. Now as a consequence of (29), we have (5). Hence the zero solution of the switched system (1) is asymptotically stable almost surely. \square

Theorem 2.3 provides an almost sure asymptotic stability condition for the switched system (1). This result indicates that the stability can be assessed by checking the inequality (17) for all scalars $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that (19), (20) hold.

In (17), the scalar $\gamma_q \in \mathbb{R}$ represents the effect of mode sequence q . Specifically, $\gamma_q < 0$ implies that the norm of the system's state gets smaller after h time steps, if the mode of the switched system within those h time steps follows the sequence q . On the other hand, $\gamma_q > 0$ may indicate an increase in the norm of the system's state. Hence, the term $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$ in (17) with $\rho_q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ would correspond to the average of the effects of all h -length sequences in \mathcal{M}^h . However, this average cannot be computed directly, since in this paper, we consider the case where the specific values of $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ are not available.

On the other hand, we show by using Lemma 2.2 that if the long-run average activity of modes is known to be bounded as in (2) and (3), then $\rho_q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ would satisfy (19), (20). Hence, for stability analysis, one can check the sign of $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$ in (17) for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, that satisfy (19), (20). This is equivalent to checking the stability for all possible mode sequence scenarios that satisfy (2) and (3), since different values of the limits $\rho_q = \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, represent different scenarios. We will show in Section III that rather than checking the condition in (17) for all possible scenarios, we can utilize linear programming methods to identify the worst scenario in terms of stability, and check the condition only for that scenario.

Note that in Theorem 2.3 we require the existence of $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ for all $q \in \mathcal{M}^h$, even though the particular values of these limits are not needed for stability analysis. The following result identifies a class of mode signals $\{r(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ for which the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist. Using this result, we will show that the limits exist in a variety of scenarios.

Proposition 2.4: Let $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ with $g(0) = g_0 \in \mathcal{S}$ be a finite-state irreducible Markov chain. Assume $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ is given by

$$r(t) \triangleq \begin{cases} 1, & g(t) \in \mathcal{S}_1, \\ \vdots & \\ M, & g(t) \in \mathcal{S}_M, \end{cases} \quad t \in \mathbb{N}_0, \quad (30)$$

where $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_M$ form a partition of the set \mathcal{S} , i.e., $\bigcup_{i=1}^M \mathcal{S}_i = \mathcal{S}$ and $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$, $i \neq j$. Then for all $h \in \mathbb{N}$, the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist.

The proof of this result is based on the observation that $\bar{r}(\cdot)$ is generated from sequences of values that the process $r(\cdot)$ takes between every h time steps. First, we construct a new process $\bar{g}(\cdot)$ representing the sequences of values that

$g(\cdot)$ takes between every d steps, where d is a suitably chosen period length. We then establish the relation between the processes $\bar{r}(\cdot)$ and $\bar{g}(\cdot)$ by using (30), and show that $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ can be obtained by utilizing invariant distributions of $\bar{g}(\cdot)$.

Proof: In the proof, we use the notion of *period* for Markov chains [47]. Specifically, the period $\tau_\sigma \in \mathbb{N}$ of a state $\sigma \in \mathcal{S}$ is defined by

$$\tau_\sigma \triangleq \gcd\{t \in \mathbb{N} : \mathbb{P}[g(t) = \sigma | g(0) = \sigma] > 0\}, \quad \sigma \in \mathcal{S},$$

where $\gcd(T)$ denotes the greatest common denominator of the elements of the set T . By this definition, the random time intervals between revisits to state σ are guaranteed to be integer multiples of τ_σ . Since $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ is an irreducible finite-state Markov chain, it follows from Corollary 8.3.7 of [47] that the period is the same for all states. We use $\tau \in \mathbb{N}$ to denote this period, i.e., $\tau = \tau_1 = \tau_2 = \dots = \tau_{|\mathcal{S}|}$, where $|\mathcal{S}|$ denotes the number of elements in the set \mathcal{S} . Now let $d \triangleq \tau h$.

Next, we define a sequence-valued process to characterize the evolution of $g(\cdot)$ in every d steps. To this end, first, for each $\sigma \in \mathcal{S}$, let $\mathcal{I}_{\sigma,k} \triangleq \{s \in \mathcal{S} : \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\}$, $k \in \{1, \dots, |\mathcal{S}|\}$, and $\mathcal{I}_\sigma \triangleq \bigcup_{k=1}^{|\mathcal{S}|} \mathcal{I}_{\sigma,k}$. The set $\mathcal{I}_\sigma \subset \mathcal{S}$ denotes the states that can be reached from the state σ in steps that are integer multiples of d . In addition, for each $\sigma \in \mathcal{S}$, let

$$\bar{\mathcal{S}}_\sigma \triangleq \{(\bar{s}_1, \bar{s}_2, \dots, \bar{s}_d) : \bar{s}_j \in \mathcal{S}, j \in \{1, \dots, d\}, \bar{s}_1 \in \mathcal{I}_\sigma, \mathbb{P}[g(1) = \bar{s}_2, \dots, g(d-1) = \bar{s}_d | g(0) = \bar{s}_1] > 0\}.$$

Now we define the sequence-valued process $\{\bar{g}(i)\}_{i \in \mathbb{N}_0}$ by

$$\bar{g}(i) \triangleq (g(id), g(id+1), \dots, g((i+1)d-1)). \quad (31)$$

Notice that $\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}$, $i \in \mathbb{N}_0$.

Our next goal is to show that the sequence-valued Markov chain $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$ is an irreducible Markov chain. Specifically, we prove that for every $\bar{\sigma}, \bar{s} \in \bar{\mathcal{S}}_{g_0}$, there exists $\bar{k} \in \mathbb{N}$ such that

$$\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | \bar{g}(i) = \bar{\sigma}] > 0. \quad (32)$$

To this end, first note that $\bar{g}_1(i) \in \mathcal{I}_{g_0}$, $i \in \mathbb{N}_0$, that is, the first elements of the sequence-values that $\bar{g}(\cdot)$ takes are elements of the set \mathcal{I}_{g_0} . It follows from the definition of \mathcal{I}_{g_0} that for all $\sigma, s \in \mathcal{I}_{g_0}$,

$$\{k \in \mathbb{N} : \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\} \neq \emptyset.$$

Now, define $k : \mathcal{I}_{g_0} \times \mathcal{I}_{g_0} \rightarrow \mathbb{N}$ by

$$k(\sigma, s) \triangleq \min\{k \in \mathbb{N} : \mathbb{P}[g(kd) = s | g(0) = \sigma] > 0\}. \quad (33)$$

Moreover, note that for any given $\bar{\sigma}, \bar{s} \in \bar{\mathcal{S}}_{g_0}$, we can always pick a state $c \in \mathcal{I}_{g_0}$ and let $\bar{k} \triangleq k(c, \bar{s}_1) + 1$ so that

$$\mathbb{P}[g(1) = c | g(0) = \bar{\sigma}_d] > 0.$$

By (33), we have $\mathbb{P}[g(k(c, \bar{s}_1)d) = \bar{s}_1 | g(0) = c] > 0$, and consequently

$$\begin{aligned}
\mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | \bar{g}(i) = \bar{\sigma}] &= \mathbb{P}[\bar{g}(i + \bar{k}) = \bar{s} | g((i + 1)d - 1) = \bar{\sigma}_d] \\
&= \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \dots, \\
&\quad g((i + \bar{k} + 1)d - 1) = \bar{s}_d | g((i + \bar{k})d) = \bar{s}_1] \\
&\quad \cdot \mathbb{P}[g((i + \bar{k})d) = \bar{s}_1 | g((i + 1)d - 1) = \bar{\sigma}_d] \\
&\geq \mathbb{P}[g((i + \bar{k})d + 1) = \bar{s}_2, \dots, \\
&\quad g((i + \bar{k} + 1)d - 1) = \bar{s}_d | g((i + \bar{k})d) = \bar{s}_1] \\
&\quad \cdot \mathbb{P}[g((i + \bar{k})d) = \bar{s}_1 | g((i + 1)d) = c] \\
&\quad \cdot \mathbb{P}[g((i + 1)d) = c | g((i + 1)d - 1) = \bar{\sigma}_d] > 0.
\end{aligned}$$

Thus, the sequence-valued Markov chain $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$ is irreducible. Now, define the function $\alpha : \bar{\mathcal{S}}_{g_0} \times \mathcal{M}^h \rightarrow \mathbb{N}_0$ by

$$\alpha(\bar{s}, q) \triangleq \sum_{j=0}^{\tau-1} \mathbb{1}[\bar{s}_{jh+1} \in \mathcal{S}_{q_1}, \dots, \bar{s}_{jh+h} \in \mathcal{S}_{q_h}], \quad (34)$$

for $\bar{s} \in \bar{\mathcal{S}}_{g_0}$, $q \in \mathcal{M}^h$. Note that $\alpha(\bar{s}, q) \in \mathbb{N}_0$ is the number of times the sequence q appears in the process $r(\cdot)$, when the process $g(\cdot)$ takes the values $\bar{s}_1, \bar{s}_2, \dots, \bar{s}_d$. This number is computed by dividing the d -length sequence \bar{s} into τ number of h -length sequences and counting the number of h -length sequences whose elements are from sets $\mathcal{S}_{q_1}, \mathcal{S}_{q_2}, \dots, \mathcal{S}_{q_h}$.

By using (34), we get

$$\frac{1}{\tau k} \sum_{i=0}^{\tau k-1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau k} \sum_{i=0}^{k-1} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \mathbb{1}[\bar{g}(i) = \bar{s}] \alpha(\bar{s}, q). \quad (35)$$

Let $\bar{\pi}_{g_0, \bar{s}} \in [0, 1]$, $\bar{s} \in \bar{\mathcal{S}}_{g_0}$, denote the invariant distribution associated with $\{\bar{g}(i) \in \bar{\mathcal{S}}_{g_0}\}_{i \in \mathbb{N}_0}$. It then follows from ergodic theorem for finite-state Markov chains (see Theorem 1.10.2 of [48]) that

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \mathbb{1}[\bar{g}(i) = \bar{s}] \alpha(\bar{s}, q) = \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q).$$

Hence, as a consequence of (35), $\lim_{k \rightarrow \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k-1} \mathbb{1}[\bar{r}(i) = q]$ exists and is given by

$$\lim_{k \rightarrow \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k-1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q). \quad (36)$$

Our final goal is to show

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] = \lim_{k \rightarrow \infty} \frac{1}{\tau k} \sum_{i=0}^{\tau k-1} \mathbb{1}[\bar{r}(i) = q].$$

To this end, first let $\theta(k) \triangleq \lfloor \frac{k}{\tau} \rfloor$ and observe that

$$\begin{aligned}
\frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] &= \frac{1}{k} \sum_{i=0}^{\tau \theta(k)-1} \mathbb{1}[\bar{r}(i) = q] \\
&\quad + \frac{1}{k} \sum_{i=\tau \theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q], \quad k \in \mathbb{N}_0.
\end{aligned} \quad (37)$$

Since $\mathbb{1}[\bar{r}(i) = q] \in \{0, 1\}$, we have $0 \leq \frac{1}{k} \sum_{i=\tau \theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \frac{\tau}{k}$, and hence, $0 \leq \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=\tau \theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] \leq \lim_{k \rightarrow \infty} \frac{\tau}{k} = 0$. As a result,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=\tau \theta(k)}^{k-1} \mathbb{1}[\bar{r}(i) = q] = 0. \quad (38)$$

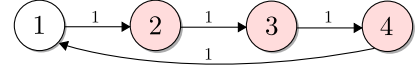


Figure 1. Transition diagram for Markov chain $\{g(t) \in \mathcal{S} = \{1, 2, 3, 4\}\}_{t \in \mathbb{N}_0}$ with initial condition $g(0) = 1$ and partitions $\mathcal{S}_1 = \{1\}$, $\mathcal{S}_2 = \{2, 3, 4\}$, characterizing a 4-periodic mode sequence $1, 2, 2, 2, 1, 2, 2, 2, \dots$

Furthermore, we have

$$\begin{aligned}
\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{\tau \theta(k)-1} \mathbb{1}[\bar{r}(i) = q] &= \lim_{k \rightarrow \infty} \frac{1}{k} \frac{\tau \theta(k)}{\tau \theta(k)} \sum_{i=0}^{\tau \theta(k)-1} \mathbb{1}[\bar{r}(i) = q] \\
&= \lim_{k \rightarrow \infty} \frac{\tau \theta(k)}{k} \lim_{k \rightarrow \infty} \frac{1}{\tau \theta(k)} \sum_{i=0}^{\tau \theta(k)-1} \mathbb{1}[\bar{r}(i) = q]. \quad (39)
\end{aligned}$$

Since $\lim_{k \rightarrow \infty} \frac{\tau \theta(k)}{k} = 1$, by (36) and (39),

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{\tau \theta(k)-1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q). \quad (40)$$

It then follows from (37), (38), and (40) that $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists and is given by

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q] = \frac{1}{\tau} \sum_{\bar{s} \in \bar{\mathcal{S}}_{g_0}} \bar{\pi}_{g_0, \bar{s}} \alpha(\bar{s}, q).$$

□

In Proposition 2.4, we provide a characterization of the mode signal $\{r(t) \in \{1, \dots, M\}\}_{t \in \mathbb{N}_0}$ through an irreducible Markov chain $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$. The set \mathcal{S} of the possible values of $g(\cdot)$ is the union of disjoint sets $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_M$. By the definition in (30), the mode signal takes the value s , when $g(t) \in \mathcal{S}_s$.

The characterization through (30) is general enough to model various types of mode signals. For instance, periodic mode switchings can be described with an irreducible and periodic Markov chain $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$. Consider for example a switched system with 2 modes. The mode sequence is assumed to repeat itself in every 4 time steps. Specifically, in every 4 time steps, mode 1 is active for 1 time step then mode 2 becomes active for the next 3 time steps. This periodic switching scenario can be characterized by setting $\mathcal{S}_1 \triangleq \{1\}$, $\mathcal{S}_2 \triangleq \{2, 3, 4\}$, and $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ as a Markov chain with transition probabilities shown on the edges of the transition graph in Fig. 1. In this situation $g(\cdot)$ repeatedly takes the values $1, 2, 3, 4, 1, 2, 3, 4, \dots$. As a result, by the definition in (30), the mode signal $r(\cdot)$ takes the values $1, 2, 2, 2, 1, 2, 2, 2, \dots$ indicating the periodic change in the mode. In this example, the switched system can be used for modeling a networked control system under periodic attacks. In particular, the first mode corresponds to a successful packet exchange between the plant and the controller, and the second mode represents the dynamics when there is a transmission failure due to attacks. The characterization of the mode signal through the setup in Fig. 1 represents an example of the discrete-time version of the periodic attacks discussed in [16]. Here the attacker periodically repeats sleeping for 1 time step and emitting a jamming signal to block network transmissions

for 3 consecutive time steps. It is important to note that when the networked control system is periodically attacked, the specific failure sequence and the period itself are part of attacker's strategy and in general they are not available to the system operator. We consider a networked control problem that covers this case in Section IV. There, we show that to check networked control system's stability through Theorem 2.3, only the knowledge of the upper-bound on the average attack ratio is needed. For the periodic attack described through the mode switching diagram in Fig. 1, the upper-bound on this ratio is given by $\bar{\rho}_2$, since the second mode corresponds to the attacks. Notice that in this case we can select $\underline{\rho}_1 = 1 - \bar{\rho}_2$, $\bar{\rho}_1 = 1$, and $\underline{\rho}_2 = 0$.

The characterization in (30) can also be used to describe random packet transmission failures. For example, communication channels following the Markov model can be described simply by setting $\mathcal{S}_1 \triangleq \{1\}$, $\mathcal{S}_2 \triangleq \{2\}$, and $\{g(t) \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2\}_{t \in \mathbb{N}_0}$ as a Markov chain with certain transition probabilities. In addition, the Gilbert-Elliott model and other more advanced models based on Markov chains (see [41], [42]) can also be described within the framework. For instance, in Gilbert-Elliott model, the channel is in the state of either "Good" or "Bad". In Good channel state, packet losses occur with a small probability e ; moreover, in Bad channel state, failure probability denoted by f may be large. Transitions between Good and Bad states occur with probability p from Good to Bad and q from Bad to Good. This scenario can be described by setting $\mathcal{S}_1 \triangleq \{1, 2\}$, $\mathcal{S}_2 \triangleq \{3, 4\}$, and $\{g(t) \in \mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2\}_{t \in \mathbb{N}_0}$ as a Markov chain with transition diagram shown in Fig. 2. In this setting, $g(t) \in \{1, 3\}$ corresponds to Good channel state and $g(t) \in \{2, 4\}$ corresponds to Bad. On the other hand, by (30), $g(t) \in \mathcal{S}_2$ indicates a packet exchange failure at time t , whereas $g(t) \in \mathcal{S}_1$ indicates a successful packet exchange attempt. Using different settings for \mathcal{S}_1 , \mathcal{S}_2 , and $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$, we can also model the situation where the network faces both malicious attacks and random packet transmission failures.

Note that when $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$ is characterized through (30), the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist for all $h \in \mathbb{N}$. Hence, in such cases, the stability analysis in Theorem 2.3 can be conducted with any $h \in \mathbb{N}$. On the other hand, for other characterizations of $\{r(t) \in \mathcal{M}\}_{t \in \mathbb{N}_0}$, it may be the case that the limits exist for $h \in \{1, 2, \dots, \hat{h}\}$ but not for $h > \hat{h}$, where $\hat{h} \in \mathbb{N}$. In those situations, Theorem 2.3 is applicable only for $h \in \{1, 2, \dots, \hat{h}\}$.

III. LINEAR PROGRAMMING METHODS FOR STABILITY ASSESSMENT

In this section we investigate two closely-related linear programming problems and present a method for checking almost sure asymptotic stability condition given in Theorem 2.3 through their optimal solutions.

A. Linear Programming Problem 1

Theorem 2.3 states that the switched system (1) is stable if there exists an induced matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0, 1)$ such that the inequality (17) holds for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$,

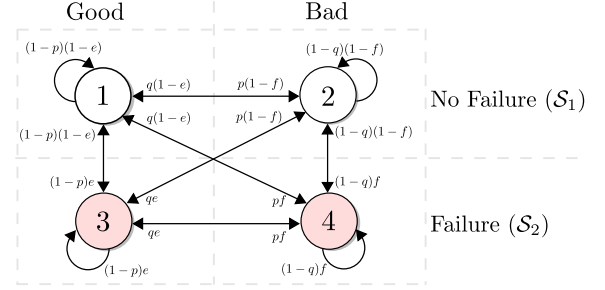


Figure 2. Transition diagram for Markov chain $\{g(t) \in \mathcal{S} = \{1, 2, 3, 4\}\}_{t \in \mathbb{N}_0}$ with partitions $\mathcal{S}_1 = \{1, 2\}$, $\mathcal{S}_2 = \{3, 4\}$, characterizing packet exchange failures on a Gilbert-Elliott channel.

that satisfy (19), (20). In what follows, we provide a linear programming problem to check this condition for a given induced matrix norm $\|\cdot\|$ and scalar $\varepsilon \in (0, 1)$.

Now define $\gamma_q \in \mathbb{R}$, $q \in \mathcal{M}^h$, as in (18) and consider the linear programming problem

$$\begin{aligned} & \text{maximize} && \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \\ & \text{subject to} && (19), (20). \end{aligned} \quad (41)$$

The optimal value of the objective function in the linear programming problem (41) corresponds to the maximum value that the summation term $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$ in (17) can take for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, that satisfy (19), (20).

For the stability analysis, different values of ρ_q , $q \in \mathcal{M}^h$, that satisfy (19), (20) represent possible mode activity scenarios such that the long run average conditions (2) and (3) hold. The linear programming problem (41) allows us to identify scenarios that maximize $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$. We can then check the stability condition (17) with the maximum value of $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$ instead of checking it for all possible scenarios.

In the following lemma, we show that the linear programming problem (41) is feasible, that is, there always exist $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, that satisfy (19), (20). Furthermore, we show that the problem is bounded (i.e., the objective function $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q$ in (41) is bounded).

Lemma 3.1: The linear programming problem (41) is feasible and bounded.

Proof: First, we show that the feasible region of the linear programming problem is not empty. To this end, first observe that $\sum_{s=1}^M \underline{\rho}_s \leq 1 \leq \sum_{s=1}^M \bar{\rho}_s$. This is because Assumption 2.1 implies

$$\begin{aligned} 1 &= \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^M \mathbb{1}[r(t) = s] \\ &\leq \sum_{s=1}^M \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \leq \sum_{s=1}^M \bar{\rho}_s, \end{aligned}$$

and similarly,

$$\begin{aligned} 1 &= \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \sum_{s=1}^M \mathbb{1}[r(t) = s] \\ &\geq \sum_{s=1}^M \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = s] \geq \sum_{s=1}^M \underline{\rho}_s. \end{aligned}$$

Now let $\underline{\rho} \triangleq \sum_{s=1}^M \underline{\rho}_s$, $\bar{\rho} \triangleq \sum_{s=1}^M \bar{\rho}_s$ and $\beta_s \triangleq (\bar{\rho}_s - \underline{\rho}_s)^{\frac{1-\underline{\rho}}{\bar{\rho}-\underline{\rho}}}$ and consider $\rho_q, q \in \mathcal{M}^h$, given by

$$\rho_q = \begin{cases} \underline{\rho}_s + \beta_s, & \text{if } c_s(q) = h, s \in \mathcal{M}, \\ 0, & \text{otherwise.} \end{cases} \quad (42)$$

To establish that the feasible region contains $\rho_q, q \in \mathcal{M}^h$, given by (42), we show that (19) and (20) hold. First, (19) holds because

$$\begin{aligned} \sum_{q \in \mathcal{M}^h} \rho_q &= \sum_{s=1}^M (\underline{\rho}_s + \beta_s) = \underline{\rho} + \frac{1-\underline{\rho}}{\bar{\rho}-\underline{\rho}} \sum_{s=1}^M (\bar{\rho}_s - \underline{\rho}_s) \\ &= \underline{\rho} \left(1 - \frac{1-\underline{\rho}}{\bar{\rho}-\underline{\rho}}\right) + \frac{1-\underline{\rho}}{\bar{\rho}-\underline{\rho}} \bar{\rho} = 1. \end{aligned} \quad (43)$$

Now we show (20). Since $\underline{\rho} \leq 1 \leq \bar{\rho}$, we have $\beta_s \in [0, \bar{\rho}_s - \underline{\rho}_s]$. It then follows that

$$\sum_{q \in \mathcal{M}^h} c_s(q) \rho_q = \underline{\rho}_s + \beta_s \in [\underline{\rho}_s, \bar{\rho}_s], \quad s \in \mathcal{M}. \quad (44)$$

It now remains to show that the solutions to the linear programming problem are bounded. Note that since $\rho_q \leq 1, q \in \mathcal{M}^h$, we have $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \leq (\max_{q \in \mathcal{M}^h} \gamma_q) \sum_{q \in \mathcal{M}^h} \rho_q \leq (\max_{q \in \mathcal{M}^h} \gamma_q) M^h < \infty$, which completes the proof. \square

A feasible and bounded linear programming problem always possesses an optimal solution (see Proposition 3.1 of [49]). Hence, Lemma 3.1 implies that there exists an optimal solution to the linear programming problem (41). Even though there may be multiple optimal solutions, we can always compute the optimal value of the objective function using any one of those solutions. Let J_h denote the optimal value of the objective function when h -length sequences are considered, that is,

$$J_h \triangleq \sup \left\{ \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q : \rho_q \in [0, 1], q \in \mathcal{M}^h, \text{ s.t. (19), (20)} \right\}.$$

The following result indicates that the stability of the switched system (1) can be assessed by checking the sign of the optimal value J_h .

Corollary 3.2: Consider the switched system (1). Suppose that the mode signal $\{r(t) \in \{1, \dots, M\}\}_{t \in \mathbb{N}_0}$ satisfies Assumption 2.1 with $\bar{\rho}_s, \underline{\rho}_s \in [0, 1], s \in \mathcal{M}$, and $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$ for a given $h \in \mathbb{N}$. The zero solution $x(t) \equiv 0$ of the switched system (1) is asymptotically stable almost surely if

$$J_h < 0. \quad (45)$$

Proof: The result is a direct consequence of Theorem 2.3, since (45) implies that (17) holds for all $\rho_q \in [0, 1], q \in \mathcal{M}^h$, that satisfy (19), (20). \square

Corollary 3.2 establishes the relation between the linear programming problem (41) and almost sure asymptotic stability of the switched system (1).

Remark 3.3: The optimal solution J_h for the linear programming problem (41) may be positive when h is small and negative for sufficiently large h . The reason is that with large h , stability/instability properties of more mode activity patterns are taken into account. For instance, consider a switched system with two modes. When $h = 2$, the effects of the dynamics associated with packet failure sequences in $\mathcal{M}^2 \triangleq$

$\{(1, 1), (1, 2), (2, 1), (2, 2)\}$ are represented by $\gamma_q, q \in \mathcal{M}^2$. However, $\gamma_q, q \in \mathcal{M}^2$, cannot be used to distinguish the difference between stabilizing (destabilizing) effects of longer mode activity sequences $(1, 2, 2, 1)$ and $(2, 1, 1, 2)$, since both of them are composed of the same smaller sequences $(1, 2), (2, 1)$. Thus to show stability, we may need to take into account longer mode sequences and obtain J_h for larger values of $h \in \mathbb{N}$.

Even though there are efficient algorithms for solving linear programming problems, it is difficult to solve (41) and obtain J_h when $h \in \mathbb{N}$ is large. This is because the number of variables $\rho_q, q \in \mathcal{M}^h$, of the problem (41) grows exponentially in h . Specifically, the number elements of the set \mathcal{M}^h , and hence the number of variables of the linear programming problem (41) is given by $f(h, M) \triangleq M^h$. In the following section, we show that an alternative linear programming problem with fewer variables shares the same optimal objective function value as that of (41). In particular, the number of variables in this alternative problem grows only polynomially in h .

B. Linear Programming Problem 2

Let $\mathcal{Z}_h \triangleq \{(z_1, z_2, \dots, z_M) : z_s \in \{0, 1, \dots, h\}, s \in \mathcal{M}, \sum_{s=1}^M z_s = h\}$, and consider the linear programming problem

$$\begin{aligned} &\text{maximize} && \sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \\ &\text{subject to} && \sum_{z \in \mathcal{Z}_h} \rho'_z = 1, \quad (a) \\ &&& \underline{\rho}_s \leq \sum_{z \in \mathcal{Z}_h} \frac{z_s}{h} \rho'_z \leq \bar{\rho}_s, s \in \mathcal{M}, \quad (b) \end{aligned} \quad (46)$$

where

$$\gamma'_z \triangleq \max_{q \in \mathcal{M}^{h,z}} \gamma_q, \quad z \in \mathcal{Z}_h, \quad (47)$$

with $\mathcal{M}^{h,z} \triangleq \{q \in \mathcal{M}^h : c_1(q) = z_1, c_2(q) = z_2, c_3(q) = z_3, \dots, c_M(q) = z_M\}, z \in \mathcal{Z}_h$.

In what follows, we first show that the objective functions of the linear programming problems (41) and (46) have the same optimal values. Using this result, we then show that the linear programming problem (46) can also be used in the stability analysis. After that we discuss about the advantage of the linear programming problem (46) over the problem (41). Specifically, we show that it is easier to solve the linear programming problem (46) because it involves fewer variables.

Lemma 3.4: The linear programming problem (46) is feasible and bounded.

Proof: The proof is similar to that of Lemma 3.1. First, we show that the feasible region of the linear programming problem (46) is not empty. To this end, consider $\rho'_z, z \in \mathcal{Z}_h$, given by

$$\rho'_z = \sum_{q \in \mathcal{M}^{h,z}} \rho_q. \quad (48)$$

with ρ_q given in (42). Notice that $\rho'_z \in [0, 1], z \in \mathcal{Z}_h$. This is because $\rho_q \geq 0, q \in \mathcal{M}^h$, implies $\rho'_z \geq 0$, and moreover, $\rho'_z = \sum_{q \in \mathcal{M}^{h,z}} \rho_q \leq \sum_{q \in \mathcal{M}^h} \rho_q = 1$ due to (43). To establish that the feasible region contains $\rho'_z, z \in \mathcal{Z}_h$, given by (48), we

show that (46a) and (46b) hold. Now, (46a) holds, because it follows from (43) that

$$\sum_{z \in \mathcal{Z}_h} \rho'_z = \sum_{z \in \mathcal{Z}_h} \sum_{q \in \mathcal{M}^{h,z}} \rho_q = \sum_{q \in \mathcal{M}^h} \rho_q = 1,$$

where we also used the fact that $\mathcal{M}^{h,z} \cap \mathcal{M}^{h,\hat{z}} = \emptyset$ for $z \neq \hat{z}$, $z, \hat{z} \in \mathcal{Z}_h$, and $\cup_{z \in \mathcal{Z}_h} \mathcal{M}^{h,z} = \mathcal{M}^h$.

Next, to show (46b), note that $\sum_{z \in \mathcal{Z}_h} \frac{z_s}{h} \rho'_z = \sum_{z \in \mathcal{Z}_h} \sum_{q \in \mathcal{M}^{h,z}} \frac{z_s}{h} \rho_q = \sum_{q \in \mathcal{M}^h} \frac{c_s(q)}{h} \rho_q$. Hence, (44) implies (46b).

It now remains to show that the solutions to the linear programming problem are bounded. Note that since $\rho'_z \leq 1$, $z \in \mathcal{Z}_h$, it follows from (47) that $\sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \leq (\max_{z \in \mathcal{Z}_h} \gamma'_z) \sum_{z \in \mathcal{Z}_h} \rho'_z \leq (\max_{q \in \mathcal{M}^h} \gamma_q) \binom{h+M-1}{M-1} < \infty$, which completes the proof. \square

By Lemma 3.4, there exists an optimal solution to the linear programming problem (46). Let J'_h denote the optimal value of the objective function for a given $h \in \mathbb{N}$, that is,

$$J'_h \triangleq \sup \left\{ \sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z : \rho'_z, z \in \mathcal{Z}_h, \text{ s.t. (46a), (46b)} \right\}.$$

Lemma 3.5: The objective functions of the linear programming problems (41) and (46) have the same optimal values, that is, $J_h = J'_h$.

Proof: We prove this result by showing $J_h \leq J'_h$ and $J_h \geq J'_h$ separately.

To establish $J_h \leq J'_h$, we show that for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that (19), (20) hold, we have $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \leq J'_h$. Now, notice that for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that (19), (20) hold, the constraints in (46a) and (46b) hold with $\rho'_z \triangleq \sum_{q \in \mathcal{M}^{h,z}} \rho_q$, $z \in \mathcal{Z}_h$, and as a result, $\sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \leq J'_h$. Hence, for all $\rho_q, q \in \mathcal{M}^h$, such that (19), (20) hold, we have

$$\begin{aligned} \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q &= \sum_{z \in \mathcal{Z}_h} \sum_{q \in \mathcal{M}^{h,z}} \gamma_q \rho_q \leq \sum_{z \in \mathcal{Z}_h} \sum_{q \in \mathcal{M}^{h,z}} \gamma'_z \rho_q \\ &= \sum_{z \in \mathcal{Z}_h} \gamma'_z \sum_{q \in \mathcal{M}^{h,z}} \rho_q = \sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \leq J'_h, \end{aligned}$$

which implies $J_h \leq J'_h$.

To prove $J_h \geq J'_h$, we now show that there exists $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q = J'_h$ and (19), (20) hold. Here, let $\hat{\rho}'_z, z \in \mathcal{Z}_h$, denote an optimal solution to the linear programming problem (46), that is,

$$\sum_{i \in \mathcal{Z}_h} \gamma'_i \hat{\rho}'_i = J'_h.$$

Now, for each $z \in \mathcal{Z}_h$, let $q^{(z)} \in \arg\max_{q \in \mathcal{M}^{h,z}} \gamma_q$ and set $\rho_{q^{(z)}} = \hat{\rho}'_z$, $\rho_q = 0$, $q \in \mathcal{M}^{h,z} \setminus \{q^{(z)}\}$. It follows that $\rho_q, q \in \mathcal{M}^h$, satisfy (19), (20); furthermore,

$$\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q = \sum_{z \in \mathcal{Z}_h} \gamma'_{q^{(z)}} \hat{\rho}'_z = J'_h.$$

This establishes that there exists $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q = J'_h$ and (19), (20) hold, which implies that $J_h \geq J'_h$. \square

The following result is a direct consequence of Lemma 3.5 and Corollary 3.2.

Corollary 3.6: Consider the switched system (1). Suppose that the mode signal $\{r(t) \in \{1, \dots, M\}\}_{t \in \mathbb{N}_0}$ satisfies Assumption 2.1 with $\bar{\rho}_s, \underline{\rho}_s \in [0, 1]$, $s \in \mathcal{M}$, and

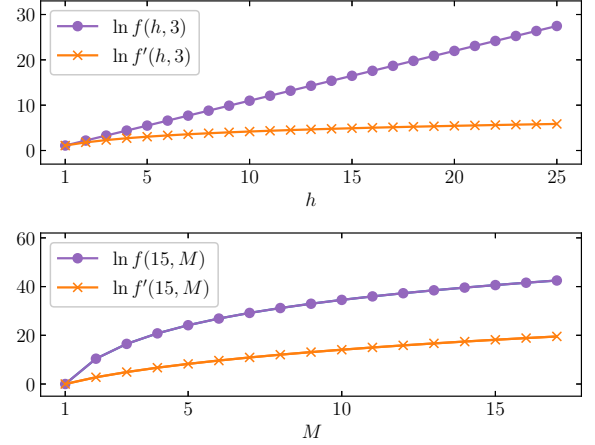


Figure 3. Comparison of $\ln f(h, M)$ and $\ln f'(h, M)$ (Top: varying h , fixed $M = 3$; Bottom: fixed $h = 15$, varying M).

$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$ exists for each $q \in \mathcal{M}^h$ for a given $h \in \mathbb{N}$. The zero solution $x(t) \equiv 0$ of the switched system (1) is asymptotically stable almost surely if $J'_h < 0$.

Corollaries 3.2 and 3.6 show that almost sure asymptotic stability of the switched system (1) can be assessed by checking signs of the optimal objective function values (J_h and J'_h) of linear programming problems (41) and (46). The switched system (1) is stable if these values are negative. Note that it is enough to compute either one of J_h or J'_h , since these scalars are shown to be equal in Lemma 3.5.

We observe that solving the linear programming problem (46) can be more advantageous in comparison to the problem (41) in terms of the computation time and the required memory. This is because the linear programming problem (46) involves fewer variables. Specifically, the number of elements of the set \mathcal{Z}_h , and hence the number of variables of the linear programming problem (46), is $f'(h, M) \triangleq \binom{h+M-1}{M-1} = \frac{(h+M-1)!}{(M-1)!h!}$. For h and M larger than 1, the number of variables in the problem (46) is strictly fewer than that of the problem (41), that is,

$$f'(h, M) < f(h, M), \quad h > 1, \quad M > 1.$$

We note again that f grows exponentially in h . On the other hand f' , the number of variables in the problem (46), grows only polynomially in h . Specifically, we have

$$f'(\alpha h, M) \leq \alpha^{M-1} f'(h, M), \quad \alpha, h, M \in \mathbb{N}.$$

As a result, when h is large, obtaining J'_h is much faster than obtaining J_h . We also remark that the number of variables grow polynomially in the number of modes M as well. In particular, we have

$$f'(h, \alpha M) \leq \alpha^h f'(h, M), \quad \alpha, h, M \in \mathbb{N}.$$

Fig. 3 shows graphs of $\ln f(h, M)$ and $\ln f'(h, M)$ indicating how the numbers of variables in problems (41) and (46) grow with respect to h and M . The difference of the numbers of variables in these two problems becomes larger as M and h gets larger. For instance, in the case where $h = 15$ and $M = 3$, there are $f(15, 3) = 14,348,907$ variables in the problem (41),

whereas the linear programming problem (46) involves only $f'(15, 3) = 136$ variables.

It is important to note that although solving the linear programming problem (46) is easier due to fewer variables, it requires precomputation of coefficients γ'_z by (47). Notice that, for each z , the complexity of computing γ'_z is linear in the number of variables of the set $\mathcal{M}^{h,z}$, which is given by $\frac{h!}{z_1!z_2!\dots z_M!}$. It turns out that this computation can be carried out for all coefficients γ'_z at the same time using parallel computing techniques. For instance, in the case of $h = 15$ and $M = 3$, the set \mathcal{Z}_h of $f'(15, 3) = 136$ variables can be partitioned into 8 subsets with size 17. We can then utilize a computer with 8 central processing units to carry out the computation for each of these subsets. Specifically, the i th processing unit computes γ'_z for all z in the i th subset. Both linear programming problems (41) and (46) require calculation of norms of matrix products in the computation of γ_q , $q \in \mathcal{M}^h$, given in (18). This calculation can also be conducted in parallel for different q values. We also note that using different matrix norms in the definition of γ_q , $q \in \mathcal{M}^h$, can be useful to check stability in the case of limited computational resources. This is because J_h and J'_h may be positive for a particular matrix norm and negative for another.

We remark that for stability analysis, the sign of J_h and J'_h can also be assessed by solving linear feasibility problems without computing the actual values of those scalars. In particular, we have $J_h \geq 0$, if and only if there exist $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, such that (19), (20), and $\sum_{q \in \mathcal{M}^h} \gamma_q \rho_q \geq 0$ hold. Similarly, we have $J'_h \geq 0$, if and only if there exist $\rho'_z \in [0, 1]$, $z \in \mathcal{Z}_h$, such that (46a), (46b), and $\sum_{z \in \mathcal{Z}_h} \gamma'_z \rho'_z \geq 0$ hold. Note that solving these feasibility problems is not necessarily faster in comparison to solving the associated linear programming problems, since the numbers of variables in these two feasibility problems are equal to those in the associated linear programming problems.

IV. APPLICATION TO NETWORKED CONTROL UNDER JAMMING ATTACKS

In this section we explore the networked control problem of a linear plant given by

$$\tilde{x}(t+1) = A\tilde{x}(t) + Bu(t), \quad \tilde{x}(0) = \tilde{x}_0, \quad t \in \mathbb{N}_0, \quad (49)$$

where $\tilde{x}(t) \in \mathbb{R}^{\tilde{n}}$ and $u(t) \in \mathbb{R}^m$ denote the state and the control input, respectively; furthermore, $A \in \mathbb{R}^{\tilde{n} \times \tilde{n}}$ and $B \in \mathbb{R}^{\tilde{n} \times m}$ are the state and input matrices, respectively. We consider two problem settings where we model the networked control system as a switched system. In the first problem, the plant and the remotely located controller exchange information packets over delay-free communication channels. In the second problem, two separate communication channels are used for transmission of the control input packets from the controller to the plant.

A. Control over Delay-Free Communication Links

First, we explore the networked control problem where at each time instant, the plant and the controller attempt to exchange state and control input packets over a communication

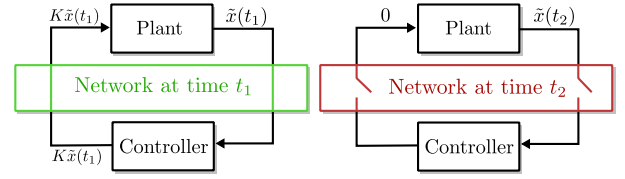


Figure 4. Packet exchange success and failure modes of the networked control system.

channel. In this problem setting, network transmissions do not face delay, but packet exchange attempts between the plant and the controller may be subject to packet losses due to malicious attacks or nonmalicious communication errors. In a successful packet exchange attempt, the plant transmits the state information to the controller; the controller uses the received state information to compute the control input through a linear control law and sends back the control input to the plant. The transmitted control input is then applied at the plant side. Packet exchange attempt failures happen when either the measured state packets or the control input packets are lost. In the case of a failed exchange attempt, the control input at the plant side is set to 0. Fig. 4 illustrates the operation of the networked control system during a successful packet exchange attempt at time t_1 and a failed exchange attempt at time t_2 .

We use the binary-valued process $\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ to describe success or failure states of packet exchange attempts. Specifically, the state $l(t) = 0$ indicates that the packet exchange attempt at time t is successful, whereas $l(t) = 1$ indicates failure. In this case, the control input $u(t)$ applied at the plant side is given by

$$u(t) \triangleq (1 - l(t)) K\tilde{x}(t), \quad t \in \mathbb{N}_0, \quad (50)$$

where $K \in \mathbb{R}^{m \times \tilde{n}}$ denotes the feedback gain.

In [24], we proposed a characterization for $\{l(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ that allows us to model the effects of random packet losses and malicious attacks in a unified manner. This characterization relies on the following assumption.

Assumption 4.1: There exists a scalar $\rho \in [0, 1]$ such that

$$\sum_{k=1}^{\infty} \mathbb{P}\left[\sum_{i=0}^{k-1} l(i) > \rho k\right] < \infty. \quad (51)$$

Here, the inequality (51) can be considered as a condition on the evolution of tail probability $\mathbb{P}\left[\sum_{i=0}^{k-1} l(i)/k > \rho\right]$. The scalar $\rho \in [0, 1]$ in (51) plays a key role in characterizing a probabilistic bound on the average ratio of packet exchange failures. Observe that the case where all packet transmission attempts result in failure can be described by setting $\rho = 1$. However in such cases, the plant cannot receive control inputs and if the uncontrolled ($u(t) \equiv 0$) dynamics are unstable, the state of the control system diverges. It is shown in [18] that ρ in (51) can be obtained to be strictly smaller than 1 for certain random and malicious packet loss models. These models include time-inhomogeneous Markov chains for describing random packet losses, as well as a discrete-time version of the malicious attack model in [17], where the number of packet exchange attempts that face attacks are upper bounded by a certain ratio of the total number of packet exchange attempts.

In the case of malicious jamming attacks, ρ also corresponds to an upper bound on average energy usage of the jammer.

We showed in [18] that the closed-loop networked control system is stable, when ρ in (51) takes sufficiently small values. In what follows we provide an alternative stability analysis method for the networked control system by utilizing Theorem 2.3. This new method allows us to show that in certain scenarios, networked control systems remain stable even for values of ρ that are larger than the ones required by the results in [18]. To utilize Theorem 2.3 we first describe the closed-loop system as a discrete-time switched system.

The networked control system (49), (50) is equivalently described as a discrete-time switched system (1) with the state $x(t) = \tilde{x}(t)$ and the mode signal given by $r(t) = l(t) + 1$. Furthermore, the subsystem matrices are given by $A_1 = A + BK$, $A_2 = A$. Now, under Assumption 4.1, the inequalities (2) and (3) in Assumption 2.1 hold with $\underline{\rho}_1 = 1 - \rho$, $\bar{\rho}_1 = 1$, $\underline{\rho}_2 = 0$, and $\bar{\rho}_2 = \rho$. This is because (51) implies

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} l(t) \leq \rho, \quad (52)$$

and hence $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[r(t) = 2] \leq \rho$. Through this switched system characterization, stability of the networked control system (49), (50) can be analyzed by using Theorem 2.3. Furthermore, the linear programming problems developed in Section III can also be used for the stability analysis.

We remark that a sufficient condition for the stability of the networked control system (49), (50) was also presented in [18] (see also [24]). There, an event-triggering controller is used for stabilization, and the packet exchange attempt times are decided by utilizing a set of triggering conditions. These conditions can be adjusted to consider the problem setting where the plant and the controller attempt packet exchanges at each time instant. For this problem setting, Theorem 2.3 can be considered as an enhancement of the stability result presented in [18]. The stability result in [18] is obtained by analyzing the evolution of a Lyapunov-like function $V(x) = x^T P x$ at each time step. This analysis idea can be recovered by our approach presented in this paper through setting $h = 1$, and defining the norm in (18) as the matrix norm induced by the vector norm $\|x\|_P \triangleq \sqrt{x^T P x}$. The following result shows that when $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i)$ exists, for all scenarios in which the stability condition in Theorem 3.5 in [18] holds, the stability condition in Theorem 2.3 also holds. This indicates that the stability condition obtained in this paper is less conservative than the stability condition obtained in [18].

Proposition 4.1: Assume $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i)$ exists. Suppose the stability condition in Theorem 3.5 in [18] holds, that is, there exist a positive-definite matrix $P \in \mathbb{R}^{n \times n}$ and scalars $\beta \in (0, 1)$, $\varphi \in [1, \infty)$ such that

$$\beta P - (A + BK)^T P (A + BK) \geq 0, \quad (53)$$

$$\varphi P - A^T P A \geq 0, \quad (54)$$

$$(1 - \rho) \ln \beta + \rho \ln \varphi < 0, \quad (55)$$

hold. Then with $h \triangleq 1$ the stability condition in Theorem 2.3 also holds, i.e., $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist;

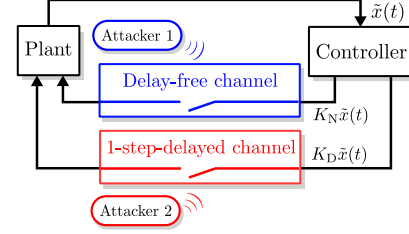


Figure 5. Networked control over delay-free and 1-step-delayed channels.

moreover, there is a matrix norm $\|\cdot\|$ and a scalar $\varepsilon \in (0, 1)$ such that (17) holds for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h$, that satisfy (19) and (20).

Proof: First, note that when $h = 1$, we have $\mathcal{M}^h = \{(1), (2)\}$. As a result, existence of $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} l(i)$ implies that $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, also exist.

Next, let the norm in (18) be the matrix norm induced by the vector norm $\|x\|_P \triangleq \sqrt{x^T P x}$, that is, $\|M\| \triangleq \sup_{x \in \mathbb{R}^n \setminus \{0\}} \frac{\|Mx\|_P}{\|x\|_P}$, $M \in \mathbb{R}^{n \times n}$. Under this norm, the inequalities (53) and (54) imply $\|A_1\| = \|A + BK\| \leq \sqrt{\beta}$ and $\|A_2\| = \|A\| \leq \sqrt{\varphi}$. Now let $\varepsilon \triangleq \sqrt{\beta}$. Since $\varepsilon = \sqrt{\beta} < \sqrt{\varphi}$, by using (55) and $\rho_{(2)} \leq \bar{\rho}_2 = \rho$, we get

$$\begin{aligned} \sum_{q \in \mathcal{M}^h} \gamma_q \rho_q &= \gamma_{(1)} \rho_{(1)} + \gamma_{(2)} \rho_{(2)} = \gamma_{(1)} (1 - \rho_{(2)}) + \gamma_{(2)} \rho_{(2)} \\ &= (1 - \rho_{(2)}) \ln \max\{\|A + BK\|, \varepsilon\} + \rho_{(2)} \ln \max\{\|A\|, \varepsilon\} \\ &\leq (1 - \rho_{(2)}) \ln \sqrt{\beta} + \rho_{(2)} \ln \sqrt{\varphi} \\ &\leq (1 - \rho) \ln \sqrt{\beta} + \rho \ln \sqrt{\varphi} = \frac{1}{2} ((1 - \rho) \ln \beta + \rho \ln \varphi) < 0, \end{aligned}$$

for all $\rho_q \in [0, 1]$, $q \in \mathcal{M}^h = \{(1), (2)\}$, such that (19) and (20) hold, which completes the proof. \square

B. Control over Delay-Free and One-Step Delayed Communication Links

We now consider the networked control system depicted in Fig. 5, where the control actions are transmitted to the plant over two separate communication channels. We assume that one of these channels faces no delay and the other one faces 1-step delay in transmissions. Investigation of a networked control setup involving multiple channels with different delays is useful for analyzing systems that incorporate multiple actuators placed at different locations. The nodes that relay the information coming from the controller to certain actuators may induce delay due to different security measures in transmission powers, encryptions, and so on. We note that the results presented in this section can also be extended to the case where there are more than two channels that may induce different amounts of delay. Our goal here is to investigate scenarios where transmissions on two channels are subject to failures due to errors in communication or (coordinated/independent) attacks from malicious agents.

In our problem setting, the plant is as given in (49). The controller receives the system state $\tilde{x}(t)$ at each time t , and computes two control inputs $K_N \tilde{x}(t)$ and $K_D \tilde{x}(t)$ that are attempted to be transmitted on the delay-free and 1-step-delayed channels, respectively. We respectively use $\{l_N(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ and $\{l_D(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ to indicate failures on

the delay-free and the one-step-delayed channels. At time t , the plant receives the control data $K_N \tilde{x}(t)$ if the transmission on the delay-free channel is successful ($l_N(t) = 0$), and $K_D \tilde{x}(t-1)$ if the transmission on the delayed channel is successful ($l_D(t) = 0$). If both channels fail ($l_N(t) = 1$, $l_D(t) = 1$), the control input at the plant side is set to 0. Furthermore, the control data $K_D \tilde{x}(t-1)$ received from the delayed channel is used only if the transmission on the delay-free channel fails ($l_N(t) = 1$, $l_D(t) = 0$). Otherwise (when $l_N(t) = 0$, $l_D(t) = 1$ or $l_N(t) = 0$, $l_D(t) = 0$), the control input at the plant side is set to $K_N \tilde{x}(t)$ received from the delay-free channel. Hence, the control input applied at the plant is given by

$$u(t) = (1-l_N(t))K_N \tilde{x}(t) + l_N(t)(1-l_D(t))K_D \tilde{x}(t-1), \quad (56)$$

for $t \geq 1$. Assuming $u(0) = 0$, the closed-loop dynamics (49), (56) can be given by

$$\begin{bmatrix} \tilde{x}(t+2) \\ \tilde{x}(t+1) \end{bmatrix} = \begin{bmatrix} A + (1-l_N(t))BK_N & l_N(t)(1-l_D(t))BK_D \\ I_{\tilde{n}} & 0 \end{bmatrix} \cdot \begin{bmatrix} \tilde{x}(t+1) \\ \tilde{x}(t) \end{bmatrix}, \quad t \in \mathbb{N}_0. \quad (57)$$

By setting

$$x(t) \triangleq \begin{bmatrix} \tilde{x}(t+1) \\ \tilde{x}(t) \end{bmatrix}, \quad r(t) \triangleq \begin{cases} 1, & l_N(t) = 0, \\ 2, & l_N(t) = 1, l_D(t) = 0, \\ 3, & l_N(t) = 1, l_D(t) = 1, \end{cases} \quad (58)$$

for $t \in \mathbb{N}_0$, the closed-loop dynamics (57) forms a switched system (1) with 3 modes represented by

$$A_1 \triangleq \begin{bmatrix} A + BK_N & 0 \\ I_{\tilde{n}} & 0 \end{bmatrix}, \quad (59)$$

$$A_2 \triangleq \begin{bmatrix} A & BK_D \\ I_{\tilde{n}} & 0 \end{bmatrix}, \quad A_3 \triangleq \begin{bmatrix} A & 0 \\ I_{\tilde{n}} & 0 \end{bmatrix}. \quad (60)$$

Concerning the communication channels in the networked control system, we assume the following.

Assumption 4.2: There exist scalars $\sigma_N, \sigma_D, \rho_N, \rho_D \in [0, 1]$ such that

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_N(t) \geq \sigma_N, \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_N(t) \leq \rho_N, \quad (61)$$

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_D(t) \geq \sigma_D, \quad \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} l_D(t) \leq \rho_D, \quad (62)$$

hold almost surely.

Note that the scalars $\sigma_N, \sigma_D, \rho_N, \rho_D$ in this assumption provide lower- and upper-bounds on the long-run average number of transmission failures on the delay-free and 1-step-delayed channels.

In the following result, we show that under Assumption 4.2, the mode signal of the switched system representing the networked control system satisfies Assumption 2.1. Specifically, we obtain $\underline{\rho}_s, \bar{\rho}_s$, $s \in \mathcal{M} = \{1, 2, 3\}$, that satisfy (2) and (3), as functions of $\sigma_N, \sigma_D, \rho_N, \rho_D$.

Lemma 4.2: Suppose (61) and (62) in Assumption 4.2 are satisfied. Then (2) and (3) in Assumption 2.1 hold with

$$\underline{\rho}_1 = 1 - \rho_N, \quad \bar{\rho}_1 = 1 - \sigma_N, \quad (63)$$

$$\underline{\rho}_2 = \max\{0, \sigma_N - \rho_D\}, \quad \bar{\rho}_2 = \min\{\rho_N, 1 - \sigma_D\}, \quad (64)$$

$$\underline{\rho}_3 = \max\{0, \sigma_N + \sigma_D - 1\}, \quad \bar{\rho}_3 = \min\{\rho_N, \rho_D\}. \quad (65)$$

Proof: We use Lemma A.1 to show the result. To this end, first note that

$$\mathbb{1}[r(t) = 1] = 1 - l_N(t), \quad (66)$$

$$\mathbb{1}[r(t) = 2] = l_N(t)(1 - l_D(t)), \quad (67)$$

$$\mathbb{1}[r(t) = 3] = l_N(t)l_D(t), \quad t \in \mathbb{N}_0. \quad (68)$$

First, we show (2) and (3) hold for the case $s = 1$ with $\underline{\rho}_1, \bar{\rho}_1$ given in (63). By (61) and (62), the inequalities (70) and (71) hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = 1 - l_N(t)$, $\xi_2(t) = 1$, $t \in \mathbb{N}_0$, and $\varsigma_1 = 1 - \rho_N$, $\varrho_1 = 1 - \sigma_N$, $\varsigma_2 = 1$, $\varrho_2 = 1$. By applying Lemma A.1, we obtain (2) from (72) and (3) from (73) with $\underline{\rho}_1, \bar{\rho}_1$ given in (63).

Next, we show (2) and (3) hold for the case $s = 2$ with $\underline{\rho}_2, \bar{\rho}_2$ given in (64). By (61) and (62), the inequalities (70) and (71) hold with $\xi_1(\cdot), \xi_2(\cdot)$ given by $\xi_1(t) = l_N(t)$, $\xi_2(t) = 1 - l_D(t)$, $t \in \mathbb{N}_0$, and $\varsigma_1 = \sigma_N$, $\varrho_1 = \rho_N$, $\varsigma_2 = 1 - \rho_D$, $\varrho_2 = 1 - \sigma_D$. By applying Lemma A.1, we obtain (2) from (72) and (3) from (73) with $\underline{\rho}_2, \bar{\rho}_2$ given in (64). The result for the other case ($s = 3$) is obtained similarly by using Lemma A.1 together with (68). \square

Lemma 4.2 shows that the networked control system (49), (56) with communication channels satisfying Assumption 4.2 can be represented by a switched system with a mode signal that satisfies Assumption 2.1. As a result, Theorem 2.3 and the linear programming problems developed in Section III can be used for the stability analysis.

V. NUMERICAL EXAMPLES

In this section, we illustrate the efficacy of our results by investigating stability properties of networked control systems discussed in Sections IV-A and IV-B.

A) Example 1: Consider the system (49) with

$$A = \begin{bmatrix} 1 & 0.1 \\ -0.5 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.1 \\ 1.2 \end{bmatrix}. \quad (69)$$

In [18], we explored stabilization of this system over a network that faces random and malicious packet losses. To guarantee stabilization, we proposed a linear state feedback controller with feedback gain $K = [-2.9012 \quad -0.9411]$. By utilizing Theorem 3.5 of [18], we see that the closed-loop system is almost surely asymptotically stable whenever the scalar ρ identified in Assumption 4.1 is inside the range $[0, 0.411]$. In the following we show that even for strictly larger values of ρ , the closed-loop system (49), (50) with above-mentioned parameters remains almost surely asymptotically stable. To see this, we employ the results provided in Sections II-B and III.

For investigating almost sure asymptotic stability of the closed-loop system (49), (50), we first characterize it as a switched system (1) with two modes represented with $A_1 = A + BK$ and $A_2 = A$. Notice that for this switched system,

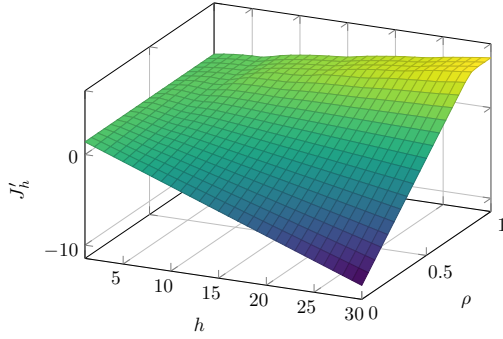


Figure 6. Optimal solution value J'_h of the linear programming problem (46) with respect to h and ρ .

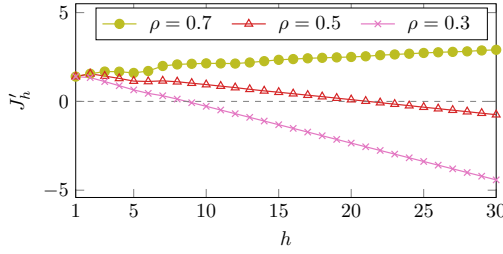


Figure 7. Optimal solution value J'_h of the linear programming problem (46) with respect to h for $\rho = 0.3$, $\rho = 0.5$ and $\rho = 0.7$.

Assumption 4.1 implies (52), and as a result, Assumption 2.1 holds with $\underline{\rho}_1 = 1 - \rho$, $\bar{\rho}_1 = 1$, $\underline{\rho}_2 = 0$, and $\bar{\rho}_2 = \rho$.

We numerically solve the linear programming problems (41) and (46) to obtain J_h and J'_h for different values of ρ and h . For finding the coefficients γ_q and γ'_z of the objective functions, we use (18) (with matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$) and (47). We numerically confirm that $J_h = J'_h$, for $h = \{1, \dots, 11\}$. For $h \geq 12$, we utilize only the linear programming problem (46) and obtain J'_h , as solving the linear programming problem (41) takes excessively long times. We see in Fig. 6 that for smaller values of ρ , J'_h takes a negative value indicating almost sure asymptotic stability. In particular, we see in Fig. 7 that when $\rho = 0.5$, we obtain $J'_{22} < 0$. It follows from Corollary 3.6 that if Assumption 4.1 holds with $\rho = 0.5$, and $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$ exists for each $q \in \mathcal{M}^{22}$, then the zero solution of the closed-loop system is almost surely asymptotically stable.

Note that for a given ρ , obtaining a nonnegative value for $J_h = J'_h$ does not necessarily imply that the system is unstable. For the same ρ , the value of J'_h may be positive for small h and negative for sufficiently large h (see Remark 3.3). For instance, in this example, J'_{10} takes a negative value for $\rho = 0.3$, but not for $\rho = 0.5$ or $\rho = 0.7$ (Fig. 7). If one can only compute J'_h up to $h = 10$ due to limited computational power, then stability for $\rho = 0.5$ cannot be concluded.

We also note that using different matrix norms in the definition of $\gamma_q, q \in \mathcal{M}^h$, given in (18) results in different trajectories for J'_h . Checking J'_h for different matrix norms can be useful to analyze stability in the case of limited computational resources. For illustration, we compute J'_h with matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$ (induced respectively by the 1-norm, the Euclidean norm, and the infinity norm of vectors), as well as the matrix norm $\|\cdot\|_P$ induced by the vector norm

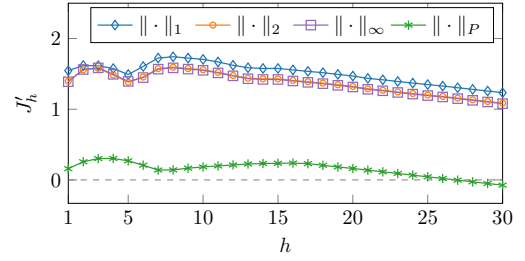


Figure 8. Optimal solution value J'_h of the linear programming problem (46) with respect to h for fixed $\rho = 0.6$ and different norms used in (18).

$\|x\|_P \triangleq \sqrt{x^T P x}$. Here, for the matrix norm $\|\cdot\|_P$, we use the positive-definite matrix P that we previously utilized in [18] for the Lyapunov-based stability analysis of this system. Fig. 8 shows the optimal solution value J'_h of the linear programming problem (46) for different values of h when $\rho = 0.6$. For this example, the values of J'_h obtained with the matrix norm $\|\cdot\|_P$ is lower than those obtained with the matrix norms $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. We also observe that J'_{30} obtained with the matrix norm $\|\cdot\|_P$ is negative. Thus we can conclude from Corollary 3.6 that if Assumption 4.1 holds with $\rho = 0.6$, and $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$ exists for each $q \in \mathcal{M}^{30}$, then the zero solution of the closed-loop system is almost surely asymptotically stable.

By calculating J'_{30} with the matrix norm $\|\cdot\|_P$ for different values of ρ , we confirm that stability is guaranteed for all $\rho \in [0, 0.6]$. Note that for $\rho \in [0, 0.6]$, the networked control system is stable under all packet exchange failure scenarios, for which the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \mathbb{1}[\bar{l}(t) = q]$, $q \in \mathcal{M}^{30}$, exist. Hence, for instance, the system is stable under all periodic attack scenarios with $\rho \leq 0.6$, since Proposition 2.4 implies that those limits exist in the periodic case. This stability region $[0, 0.6]$ is quite tight, as there exists a *destabilizing* attack strategy for $\rho = 0.64$. This attack strategy is periodic with a period of 150 time steps. It was identified through the solution $\rho_z, z \in \mathcal{Z}_{30}$, to the linear programming problem (46) solved with $h = 30$ by using the matrix norm $\|\cdot\|_2$ and the bounds $\underline{\rho}_1 = 1 - \rho$, $\bar{\rho}_1 = 1$, $\underline{\rho}_2 = 0$, $\bar{\rho}_2 = \rho$, where $\rho = 0.64$. Under this attack strategy, the mode signal of the associated switched system (1) repeats the same pattern in every 150 time steps. This pattern is composed of a particular sequence $\hat{q} \in \mathcal{M}^{30}$ appearing once and then another sequence $\hat{q} \in \mathcal{M}^{30}$ appearing 4 times. Note that under this attack strategy, Assumption 4.1 holds with $\rho = 0.64$, and furthermore, the monodromy matrix associated with the closed-loop periodic networked control system possesses an eigenvalue that is outside the unit circle of the complex plane indicating divergence of the state.

B) Example 2: In this example we demonstrate the results for the networked control setup discussed in Section IV-B. Specifically, we consider the plant with A and B given by (69) in the previous subsection. The control packets are assumed to be transmitted to the plant over the delay-free and the 1-step-delayed channels depicted in Fig. 5. The feedback gains associated with these channels are given by $K_N = \begin{bmatrix} -2.9012 & -0.9411 \end{bmatrix}$ and $K_D = \begin{bmatrix} -0.04 & -0.3 \end{bmatrix}$. We note that K_N is the feedback gain considered in the previous subsection, and K_D ensures that A_2 is a Schur matrix. The

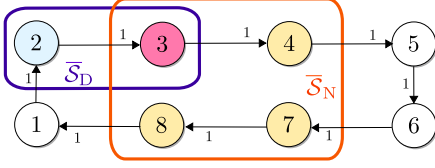


Figure 9. Transition diagram for $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ representing an 8-periodic jamming attack strategy against the networked control system depicted in Fig 5. With initial condition $g(0) = 1$, the delay-free channel is periodically attacked at times $8t+2, 8t+3, 8t+6, 8t+7$, and moreover, the 1-step-delayed channel is periodically attacked at times $8t+1, 8t+2$, for $t \in \mathbb{N}_0$.

closed-loop system (49), (56) can be represented as a switched system (1) with 3 modes described by matrices A_1, A_2, A_3 given in (59), (60) and the mode signal given by (58).

We consider the case where the delay-free and the 1-step-delayed channels are subject to coordinated periodic jamming attacks. In this case, $\{l_N(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ and $\{l_D(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$, the failure indicators of the delay-free and the one-step-delayed channels, can be given by

$$l_N(t) = \begin{cases} 1, & g(t) \in \overline{\mathcal{S}}_N, \\ 0, & \text{otherwise,} \end{cases} \quad l_D(t) = \begin{cases} 1, & g(t) \in \overline{\mathcal{S}}_D, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ is a finite-state irreducible and periodic Markov chain with transition probabilities either 0 or 1, and moreover, $\overline{\mathcal{S}}_N$ and $\overline{\mathcal{S}}_D$ are subsets of \mathcal{S} . Fig. 9 shows the transition diagram of $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$ for an example scenario where the jamming attacks follow an 8-periodic pattern. At time t , the transmission on the delay-free channel is attacked if $g(t) \in \overline{\mathcal{S}}_N = \{3, 4, 7, 8\}$; moreover, the transmission on the 1-step-delayed channel is attacked if $g(t) \in \overline{\mathcal{S}}_D = \{2, 3\}$, and both channels are attacked when $g(t) = 3$. Notice that the sets \mathcal{S} , $\overline{\mathcal{S}}_N$, and $\overline{\mathcal{S}}_D$ can be selected differently to describe more complicated attack patterns. We also remark that \mathcal{S} , $\overline{\mathcal{S}}_N$, and $\overline{\mathcal{S}}_D$ describe the attackers' strategy for the timings of the attacks and their characterization is unknown to the system operator. However, note that for any attack strategy represented with an irreducible $\{g(t) \in \mathcal{S}\}_{t \in \mathbb{N}_0}$, the mode signal given by (58) satisfies (30) with $\mathcal{S}_1 \triangleq \mathcal{S} \setminus \overline{\mathcal{S}}_N$, $\mathcal{S}_2 \triangleq \overline{\mathcal{S}}_N \cap (\mathcal{S} \setminus \overline{\mathcal{S}}_D)$, and $\mathcal{S}_3 \triangleq \overline{\mathcal{S}}_N \cap \overline{\mathcal{S}}_D$. Hence, it follows from Proposition 2.4 that for all $h \in \mathbb{N}$, the limits $\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mathbb{1}[\bar{r}(i) = q]$, $q \in \mathcal{M}^h$, exist.

To investigate the stability of the networked control system, we consider different scenarios where the long-run-average transmission failures on the delay-free and the 1-step-delayed channels satisfy Assumption 4.2. Under this assumption, Lemma 4.2 implies that the switched system representation satisfies Assumption 2.1 with $\underline{\rho}_s, \bar{\rho}_s$, $s \in \{1, 2, 3\}$, given by (63)–(65) as functions of $\sigma_N, \sigma_D, \rho_N, \rho_D$.

We first consider the case where the information about the average number of failures on delay-free and 1-step-delayed channels is limited. In particular, we explore the situation where the lower-bounds on the long-run average number of failures on both channels are 0, that is, $\sigma_N = \sigma_D = 0$. Our goal is to identify upper-bounds on the long-run average number of failures of the channels (ρ_N and ρ_D) for which the closed-loop networked control system is stable. To this end, we solve the linear programming problem (46) for different values of ρ_N and ρ_D . For finding the coefficients $\gamma'_z = \max_{q \in \mathcal{M}^h, z} \gamma_q$,

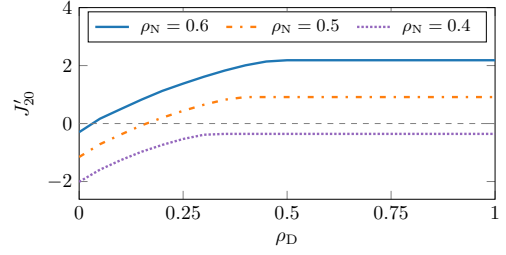


Figure 10. Optimal solution value J'_{20} of the linear programming problem (46) with respect to ρ_D for the cases $\rho_N = 0.4$, $\rho_N = 0.5$, and $\rho_N = 0.6$. (We set $\sigma_D = \sigma_N = 0$ for all cases.)

$z \in \mathcal{Z}_h$, of the objective function, we use (18) (with matrix norm induced by the Euclidean norm and $\varepsilon = 10^{-24}$). For calculating matrix norm induced by the Euclidean norm. Notice that the scalars $\sigma_N, \sigma_D, \rho_N$, and ρ_D appear in the constraints of the problem (46) according to their relation to $\underline{\rho}_s, \bar{\rho}_s$, $s \in \{1, 2, 3\}$, given in (63)–(65).

Fig. 10 shows how the optimal objective function J'_{20} of the linear programming problem (46) changes with respect to ρ_D for the values $\rho_N = 0.4, \rho_N = 0.5$, and $\rho_N = 0.6$. Observe that when the long-run average number of failures on the delay-free communication channel is sufficiently small, stability can be achieved regardless of the amount of transmission failures on the 1-step-delayed channel. This is seen in Fig. 10 for the case $\rho_N = 0.4$. Specifically, we have $J'_{20} < 0$, and hence the stability can be concluded by Corollary 3.6 for all $\rho_D \in [0, 1]$. On the other hand, when the delay-free channel faces more failures ($\rho_N = 0.5$ or $\rho_N = 0.6$), the transmissions on the 1-step-delayed channel becomes more important for achieving stability. In particular for $\rho_N = 0.5$ and $\rho_N = 0.6$, the value of J'_{20} is negative and the stability can be concluded by Corollary 3.6 *only* for sufficiently small values of ρ_D .

Next, we consider the situation where the delay-free channel is known to be completely blocked due to jamming attacks at each time instant. To explore this case, we set $\sigma_N = \rho_N = 1$. Notice that this is different from the case where $\sigma_N = 0$ and $\rho_N = 1$. In particular, the blocking information about the delay-free channel reduces the number of possible attack strategies that we need to consider for analyzing stability. More specifically, with $\sigma_N = \rho_N = 1$, the constraints in the linear programming problem (46) describes a smaller region for ρ_z , $z \in \mathcal{Z}_h$, indicating fewer strategies to be considered. In this setup, all control packets are to be transmitted over the 1-step-delayed channel. We would like to find out the long-run average number of failures that can be tolerated on this channel. For this purpose, we solve the linear programming problem (46) to obtain J'_h with respect to h for three different cases: $\rho_D = 0.1, \rho_D = 0.2$, and $\rho_D = 0.3$. In all cases we set $\sigma_D = 0, \sigma_N = \rho_N = 1$. Fig. 11 shows J'_h for different values of ρ_D . We observe that $J'_{14} < 0$ when $\rho_D = 0.1$. Hence, by Corollary 3.6 the zero solution of the closed-loop networked control system is almost surely asymptotically stable if the ratio of the transmission failures on the 1-step-delayed channel is bounded by 0.1 in the long-run. On the other hand, for $\rho_D = 0.2$ and $\rho_D = 0.3$, we cannot guarantee stability, since we have $J'_h > 0$, $h \in \{1, \dots, 20\}$. We remark that the scalars ρ_q , $q \in \mathcal{M}^h$, and ρ_z , $z \in \mathcal{Z}_h$, that are associated with the

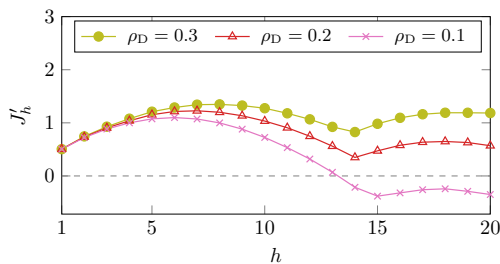


Figure 11. Optimal solution value J'_h of the linear programming problem (46) with respect to h for the cases $\rho_D = 0.1$, $\rho_D = 0.2$, and $\rho_D = 0.3$. (We set $\sigma_D = 0$, $\sigma_N = \rho_N = 1$ for all cases.)

optimal values J_h and J'_h provide useful information about damaging attack patterns and packet loss scenarios, even when stability cannot be concluded.

VI. CONCLUSION

We explored almost sure asymptotic stability of a stochastic switched linear system. Our proposed stability analysis approach relies on studying the switched system's state at every $h \in \mathbb{N}$ steps. We obtained sufficient stability conditions and showed that the stability can be checked by solving a linear programming problem. The number of variables in this problem grows polynomially in the number of subsystems, but exponentially in h , which makes the computation difficult when h is large. To overcome this issue, we also constructed an alternative linear programming problem, where the number of variables grows polynomially in the number of subsystems and polynomially in h . Even though the calculation of the coefficients in the alternative problem takes additional time, the solution is obtained faster compared to the original problem.

Our linear programming-based analysis approach allows us to check stability without relying on statistical information on the mode signal. In particular, the probability of mode switches and the stationary distributions associated with the modes are not needed for stability analysis. In this paper, we applied our approach in exploring networked control systems under malicious jamming attacks. The technical challenge in the networked control problem is that the attackers' specific strategies are not available for analysis. By using our stability analysis approach, we showed that a network control system's stability can be guaranteed under all possible attack strategies for which the long-run average number of network transmission failures satisfies certain conditions.

The analysis approach of this paper can be extended for checking the instability of a switched system. In particular, a linear programming problem can be used for assessing whether there exist mode switching scenarios that cause the system state to diverge. Our results for the instability analysis will be reported in a future work. The investigation of the case with noisy dynamics is also part of our future extensions.

REFERENCES

- [1] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Control Syst.*, vol. 19, no. 5, pp. 59–70, 1999.
- [2] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Trans. Autom. Control*, vol. 54, no. 2, pp. 308–322, 2009.

- [3] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: A Lie-algebraic condition," *Syst. Control Lett.*, vol. 37, no. 3, pp. 117–122, 1999.
- [4] J. Daafouz, P. Riedinger, and C. Iung, "Stability analysis and control synthesis for switched systems: A switched Lyapunov function approach," *IEEE Trans. Autom. Control*, vol. 47, no. 11, pp. 1883–1887, 2002.
- [5] D. Liberzon, *Switching in Systems and Control*. Birkhauser, 2003.
- [6] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. IEEE Conf. Dec. Contr.*, pp. 2655–2660, 1999.
- [7] G. Zhai, H. Guisheng, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach," *Int. J. Syst. Sci.*, vol. 32, no. 8, pp. 1055–1061, 2001.
- [8] H. Zhang, D. Xie, H. Zhang, and G. Wang, "Stability analysis for discrete-time switched systems with unstable subsystems by a mode-dependent average dwell time approach," *ISA Trans.*, vol. 53, no. 4, pp. 1081–1086, 2014.
- [9] H. Ishii, T. Basar, and R. Tempo, "Randomized algorithms for synthesis of switching rules for multimodal systems," *IEEE Trans. Autom. Control*, vol. 50, no. 6, pp. 754–767, 2005.
- [10] A. R. Teel, A. Subbaraman, and A. Sferlazza, "Stability analysis for stochastic hybrid systems: A survey," *Automatica*, vol. 50, no. 10, pp. 2435–2456, 2014.
- [11] P. Shi and F. Li, "A survey on Markovian jump systems: Modeling and design," *Int. J. Control Autom. Syst.*, vol. 13, no. 1, pp. 1–16, 2015.
- [12] Y. Fang, K. Loparo, and X. Feng, "Stability of discrete time jump linear systems," *J. Math. Systems Estim. Control*, vol. 5, no. 3, pp. 275–321, 1995.
- [13] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, *Discrete-Time Markov Jump Linear Systems*. Springer, 2004.
- [14] P. Bolzern, P. Colaneri, and G. De Nicolao, "Markov jump linear systems with switching transition rates: Mean square stability with dwell-time," *Automatica*, vol. 46, no. 6, pp. 1081–1088, 2010.
- [15] P. Bolzern, P. Colaneri, and G. De Nicolao, "Design of stabilizing strategies for discrete-time dual switching linear systems," *Automatica*, vol. 69, pp. 93–100, 2016.
- [16] H. Shisheh Foroush and S. Martínez, "On single-input controllable linear systems under periodic DoS jamming attacks," in *Proc. SIAM Conf. Contr. Appl.*, 2013.
- [17] C. De Persis and P. Tesi, "Input-to-state stabilizing control under denial-of-service," *IEEE Trans. Autom. Control*, vol. 60, no. 11, pp. 2930–2944, 2015.
- [18] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Networked control under random and malicious packet losses," *IEEE Trans. Autom. Control* (to appear), 2017. <http://dx.doi.org/10.1109/TAC.2016.2612818>.
- [19] W. Xu, W. Trappe, Y. Zhang, and T. Wood, "The feasibility of launching and detecting jamming attacks in wireless networks," in *Proc. 6th ACM Int. Symp. Mobile Ad Hoc Network. Comput.*, pp. 46–57, 2005.
- [20] S. Amin, A. A. Cárdenas, and S. S. Sastry, "Safe and secure networked control systems under Denial-of-Service attacks," in *Proc. 12th HSCC*, pp. 31–45, 2009.
- [21] K. Pelechrinis, M. Iliofotou, and S. V. Krishnamurthy, "Denial of service attacks in wireless networks: The case of jammers," *IEEE Commun. Surveys Tuts.*, vol. 13, no. 2, pp. 245–257, 2011.
- [22] A. T. Mizrak, S. Savage, and K. Marzullo, "Detecting malicious packet losses," *IEEE Trans. Parallel Distrib. Syst.*, vol. 20, no. 2, pp. 191–206, 2009.
- [23] T. Shu and M. Krunz, "Privacy-preserving and truthful detection of packet dropping attacks in wireless ad hoc networks," *IEEE Trans. Mobile Computing*, vol. 14, no. 4, pp. 813–828, 2015.
- [24] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered control over unreliable networks subject to jamming attacks," in *Proc. IEEE Conf. Dec. Contr.*, pp. 4818–4823, 2015.
- [25] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Event-triggered output feedback control resilient against jamming attacks and random packet losses," in *Proc. IFAC NecSys*, pp. 270–275, 2015.
- [26] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Random and malicious packet transmission failures on multi-hop channels in networked control systems," in *Proc. IFAC NecSys*, pp. 49–54, 2016.
- [27] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Wireless control under jamming attacks with bounded average interference power," in *Proc. IFAC World Cong.* (to appear), 2017.
- [28] J. P. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *Proc. IEEE*, vol. 95, no. 1, pp. 138–172, 2007.

- [29] H. Ishii, "Limitations in remote stabilization over unreliable channels without acknowledgements," *Automatica*, vol. 45, no. 10, pp. 2278–2285, 2009.
- [30] S. Hu and W.-Y. Yan, "Stability robustness of networked control systems with respect to packet loss," *Automatica*, vol. 43, no. 7, pp. 1243–1248, 2007.
- [31] V. Gupta, N. C. Martins, and J. S. Baras, "Optimal output feedback control using two remote sensors over erasure channels," *IEEE Trans. Autom. Control*, vol. 54, no. 7, pp. 1463–1476, 2009.
- [32] K. Okano and H. Ishii, "Stabilization of uncertain systems with finite data rates and Markovian packet losses," *IEEE Trans. Control Netw. Syst.*, vol. 1, no. 4, pp. 298–307, 2014.
- [33] A. Polanski, "Lyapunov function construction by linear programming," *IEEE Trans. Autom. Control*, vol. 42, no. 7, pp. 1013–1016, 1997.
- [34] X. Liu, "Stability analysis of switched positive systems: A switched linear copositive Lyapunov function method," *IEEE Trans. Circuits Syst. II, Exp. Briefs.*, vol. 56, no. 5, pp. 414–418, 2009.
- [35] R. Baier, L. Grüne, and S. Hafstein, "Linear programming based Lyapunov function computation for differential inclusions," *Discrete Contin. Dyn. Syst. Ser. B*, vol. 17, no. 1, pp. 33–56, 2012.
- [36] S. Zhu, Q.-L. Han, and C. Zhang, "H-gain performance analysis and positive filter design for positive discrete-time Markov jump linear systems: A linear programming approach," *Automatica*, vol. 50, no. 8, pp. 2098–2107, 2014.
- [37] P. Bolzern, P. Colaneri, and G. De Nicolao, "On almost sure stability of discrete-time Markov jump linear systems," in *Proc. IEEE Conf. Dec. Contr.*, pp. 3204–3208, 2004.
- [38] Y. Huang, J. Luo, T. Huang, and M. Xiao, "The set of stable switching sequences for discrete-time linear switched systems," *J. Math. Anal. Appl.*, vol. 377, no. 2, pp. 732–743, 2011.
- [39] J.-W. Lee and G. E. Dullerud, "Uniform stabilization of discrete-time switched and Markovian jump linear systems," *Automatica*, vol. 42, no. 2, pp. 205–218, 2006.
- [40] M. Philippe, R. Essick, G. E. Dullerud, and R. M. Jungers, "Stability of discrete-time switching systems with constrained switching sequences," *Automatica*, vol. 72, pp. 242–250, 2016.
- [41] P. Sadeghi, R. A. Kennedy, P. B. Rapajic, and R. Shams, "Finite-state Markov modeling of fading channels," *IEEE Signal Process. Mag.*, vol. 25, no. 5, pp. 57–80, 2008.
- [42] M. Ellis, D. P. Pazaros, T. Kypraios, and C. Perkins, "A two-level Markov model for packet loss in UDP/IP-based real-time video applications targeting residential users," *Comput. Netw.*, vol. 70, no. 9, pp. 384–399, 2014.
- [43] A. Cetinkaya, H. Ishii, and T. Hayakawa, "Enhanced stability analysis for networked control systems under random and malicious packet losses," in *Proc. IEEE Conf. Dec. Contr.*, pp. 2721–2726, 2016.
- [44] A. Hassibi, S. P. Boyd, and J. P. How, "Control of asynchronous dynamical systems with rate constraints on events," in *Proc. IEEE Conf. Dec. Contr.*, pp. 1345–1351, 1999.
- [45] W. Zhang, M. S. Branicky, and S. M. Phillips, "Stability of networked control systems," *IEEE Contr. Syst. Mag.*, vol. 21, no. 1, pp. 84–99, 2001.
- [46] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge University Press, 1985.
- [47] J. S. Rosenthal, *A First Look at Rigorous Probability Theory*. World Scientific Publishing, 2011.
- [48] J. Norris, *Markov Chains*. Cambridge University Press, 2009.
- [49] B. Korte and J. Vygen, *Combinatorial Optimization*. Springer, 2012.

APPENDIX

Lemma A.1: For all binary-valued processes $\{\xi_1(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ and $\{\xi_2(t) \in \{0, 1\}\}_{t \in \mathbb{N}_0}$ that satisfy

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_i(t) \geq \varsigma_i, \quad (70)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_i(t) \leq \varrho_i, \quad i \in \{1, 2\}, \quad (71)$$

almost surely with $\varsigma_i, \varrho_i \in [0, 1]$, $i \in \{1, 2\}$, we have

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) \geq \max\{0, \varsigma_1 + \varsigma_2 - 1\}, \quad (72)$$

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) \leq \min\{\varrho_1, \varrho_2\}, \quad (73)$$

almost surely.

Proof: To show (72), first note that

$$\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) = 1 - \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_1(t) \xi_2(t)). \quad (74)$$

For all $i, j \in \{1, 2\}$, $i \neq j$, we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_i(t) \xi_j(t)) \\ &= \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t)) + (1 - \xi_i(t))) \\ &\leq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \\ &\quad + \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_i(t)). \end{aligned} \quad (75)$$

Since $\xi_i(t)(1 - \xi_j(t)) \leq \xi_i(t)$ and $\xi_i(t)(1 - \xi_j(t)) \leq 1 - \xi_j(t)$, by (71), we have

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \leq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_i(t) \leq \varrho_i,$$

and by (70), we have

$$\begin{aligned} \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) &\leq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_j(t)) \\ &= 1 - \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_j(t) \leq 1 - \varsigma_j. \end{aligned}$$

Therefore,

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (\xi_i(t)(1 - \xi_j(t))) \leq \min\{\varrho_i, 1 - \varsigma_j\}. \quad (76)$$

Furthermore, since $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} (1 - \xi_i(t)) \leq 1 - \varsigma_i$, it follows from (74)–(76) that

$$\begin{aligned} \liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) &\geq 1 - (\min\{\varrho_i, 1 - \varsigma_j\} + 1 - \varsigma_i) \\ &= \varsigma_i - \min\{\varrho_i, 1 - \varsigma_j\} = \max\{\varsigma_i - \varrho_i, \varsigma_i + \varsigma_j - 1\}. \end{aligned} \quad (77)$$

Now, noting that $\varsigma_i - \varrho_i \leq 0$, $i \in \{1, 2\}$, and $\liminf_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) \geq 0$, almost surely, we have (72) from (77).

Next, we prove (73). Since $\xi_1(t) \xi_2(t) \leq \xi_1(t)$ and $\xi_1(t) \xi_2(t) \leq \xi_2(t)$, we obtain

$$\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) \leq \limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_i(t), \quad (78)$$

for $i \in \{1, 2\}$. It then follows from (71) that $\limsup_{k \rightarrow \infty} \frac{1}{k} \sum_{t=0}^{k-1} \xi_1(t) \xi_2(t) \leq \varrho_i$, $i \in \{1, 2\}$, almost surely, which then implies (73). \square